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Tilburg University

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Katholieke Universiteit Brabant, op gezag van de rector magnificus, prof. dr. L.F.W. de Klerk, in het openbaar te verdedigen ten overstaan van een door het college van decanen aangewezen commissie in de aula van de Universiteit op

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JURRIAAN BEREND JANSEN

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PROMOTOR: Prof. dr. F.A. van der Duyn Schouten

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Jorg Jansen
Tilburg, July 1998

Contents

Chapter 1

Introduction

1.1 Overview

This thesis is concerned with the mathematical analysis of a class of stochastic batching models subject to a delay-limit. Hence the key elements of all models considered here are

- "stochastic": "customer" arrivals are governed by a stochastic process;
- "batching": "customers" can be grouped to obtain economies of scale;
- "delay-limit": all "customers" must be served within a prespecified delay-limit.

The meaning of the word "customer" is not restricted to a human being who exercises demand for goods or service, but includes a wide range of other interpretations depending on the context of the model; possible examples are: a call for service, an order, a job in a computer network, or a maintenance task. The delay-limit is the maximum allowable timespan between arrival and service of a customer, i.e., the maximum time a customer is prepared to wait. When the waiting time of a customer reaches the delay-limit, we say that the customer "reaches his delay-limit" or that "his delay-limit expires". Throughout this thesis we are primarily concerned with the case of a constant and prespecified delay-limit D, which is the same for all customers. A delay-limit occurs in situations where a fixed deadline must be met, e.g., by a service contract or due to customer impatience.

The thesis is divided into two parts:

Part I – The service model;

Part II – Inventory models with a delay-limit on backorders.

In Part I we analyse a basic model, the service model, that provides the building blocks for the models in Part II. The features of the service model are

- A stochastic customer arrival process: an i.i.d. process $\{X_n,\ n = 1, 2, \ldots\}$ for a discrete-time formulation of the model (Chapter 2), and a Poisson process $\{N(t),\ t \geq 0\}$ for a continuous-time formulation of the model (Chapter 3).
- Service for any customer must commence within the delay-limit D.

- At any decision epoch the service provider can choose between two service modes:
  - Batch service: all customers present are served jointly and the system is cleared; the costs associated with a batch service consist of a fixed component (a_B) and a component that is proportional to the size of the batch (b_B);
  - Individual service: when one or more customers reach their delay-limit and no batch service is done, these customers must be given individual service against a fixed cost per customer (b_I).
- No capacity restrictions on the number of simultaneous batch and individual services, nor on the size of a batch.

In Part II we shift our attention from service to the production of exchangeable items, thereby introducing the possibility of building up serviceable inventory in anticipation of future demand. In this production/inventory setting the delay-limit constraint takes the form of a time-limit on backorders and individual services correspond to lost sales, i.e., demand can be backordered until the delay-limit expires and is "lost" thereafter. In fact, "lost" can also be interpreted as "satisfied by other means" or "contracted out"; the only relevant characteristic for the model is that this demand is removed from the system against a fixed cost per item. In order to satisfy incoming demand within the delay-limit, either serviceable inventory must be positive or a new production batch must become available before the delay-limit expires. Clearly the delay-limit must now include the production time, as opposed to the service model where the delay-limit does not include the service time. As a result, the production lead time becomes an important model variable and, like with the delay-limit, we are primarily concerned with the case of a constant production lead time L that is independent of the batch size. The relationship between the service model of Part I and the production/inventory model of Part II is summarized in Table 1.1.

Part I	Part II
service model	(production/)inventory model
demand for service	demand for a single item
customers	requested items
no serviceable inventory	serviceable inventory
delay-limit	time-limit on backorders
service time	production lead time
batch service	production batch
individual service	lost sales

Table 1.1: Relationship between Part I and Part II

The two key model parameters D (delay-limit) and L (production lead time) give rise to two fundamentally different cases, namely $D \leq L$ and $D > L$. If $D \leq L$ then it is not possible to satisfy demand from a production batch that is started after the demand has arrived, whereas if $D > L$ then demand can be satisfied from a production batch that is started within time $D-L$ from its arrival (i.e., "production to order" is possible). The case $D \leq L$ leads to an inventory-type model; this also includes the case $D = 0$ that is equivalent to a lost-sales inventory model with order lead time L. On the other hand, the case $D > L$ is more related to the service model of Part I, in the sense that a queue of waiting demand (customers) may build up. However, whereas in the service model service can only start after customer arrival (by definition), in the production/inventory model with $D > L$ production can start both before and after demand arrival.

In the production/inventory setting it is natural to think of capacity restrictions on the number and size of the production batches. In Chapter 4 we introduce a general framework for production/inventory models with a time-limit on backorders, including parameters for the maximum number of simultaneous batches (N) as well as the maximum batch size (M). Here it is assumed that the whole production batch becomes serviceable at the end of the production lead time (L), which explains the link to inventory models with a positive order lead time. As a matter of fact, the resulting model can also be classified as an inventory model with an order lead time of L, a time-limit of D on backorders, at most N orders outstanding and a maximum order size of M. In Chapters 5 and 6 we focus on the dichotomy between $D \leq L$ and $D > L$, while in Chapter 7 we make a first step towards the analysis of capacitated models.

The rest of this chapter is organized as follows. In section 1.2 we describe an application of the service model, concerning repair of failed computers within an agreed delay-limit, that stimulated our interest in the subject and laid the foundations for the underlying thesis. We also provide another illustrative example regarding the delivery of cars overseas subject to a delay-limit. In section 1.3 we give a detailed description of the service model of Part I, first the discrete-time formulation (subsection 1.3.1) and next the continuous-time formulation (subsection 1.3.2). In section 1.4 we discuss the relationship between the service model and a number of other models from the literature. We close this introductory chapter in section 1.5 with some preliminaries from Markov decision theory that will be used throughout this thesis.

1.2 Background

This research was largely motivated by the following case study (see [Szczerba 1990]). A producer of personal computers offers customers a service contract guaranteeing repair of failed computer equipment during the warranty period within four working days. In particular for major clients the penalty for exceeding the deadline can be rather substantial. A repair operation basically consists of three phases: first the pick-up of the failed equipment at the customer's premises, next the actual repair at the producer's repair facility, and finally the delivery of the overhauled equipment. We focus attention on the first phase of this process: the collection of failed equipment from various customers. When a call for service arrives it might be worthwhile to wait for more calls from the same area and collect those simultaneously in a single roundtrip. In this way a substantial reduction in transportation costs can be achieved. However, waiting too long in starting the collection would jeopardize meeting the deadlines in the service contract. In view of this threat, it has been decided that each call has to be collected within a prespecified time interval from call arrival, the delay-limit. The collection process is organized in such a way that customers can be served at any time (during working hours) in a batch of any size. The latter reflects the fact that the collection vehicles have enough capacity to collect any reasonable number of failed units in a single trip. Also, the number of vehicles that are available is large enough to start a collection whenever this would be required. The costs associated with a collection are assumed to consist of a fixed set-up cost as well as (possibly) a variable cost proportional to the batch size. However, as soon as the delay-limit of a customer expires, the customer will be served on an individual basis at a relatively high cost as compared to the proportional cost per customer in a batch service. In the example of the producer of PC's a batch service is carried out using a vehicle of the producer's own fleet, while an individual service is taken care of by an external carrier who always bills on an individual basis.

Another illustrative example is a far-east car manufacturer who sends cars to customers in Europe and is subject to a maximal order lead time (the delay-limit). The manufacturer can choose between two modes of transport: cars can be delivered by boat or by aircraft. Using aircraft has the obvious advantage that it is faster, but it involves a high variable cost per car. On the other hand, a shipment involves a certain fixed cost and a relatively low variable cost per car. Therefore the manufacturer wants to include as many cars as possible in a shipment without violating the lead-time constraint. If at some moment there are orders for which the delay-limit is about to expire while the total number of orders is still too small for a shipment to be cost-effective, then the shipment is postponed and aircraft is used only for those "urgent" deliveries. When exactly it is cost-effective to trigger the shipment is the main question that will be addressed in Part I of this thesis.

1.3 The service model

Both the example of the PC producer and the example of the far-east car manufacturer satisfy the characteristics of the service model. In a natural manner the remoteness of customers suggests order aggregation through batch services. The car manufacturer receives demands for cars, and at some point makes a mass delivery by boat. The PC producer receives calls for service, and at some point sends out a service truck for a journey to visit these customers.

Chapter 2 is devoted to a discrete-time formulation of the service model, where decisions can only be taken at discrete points in time. The main results of this chapter have been published in [Berg et al. 1998], where the delivery example is used. In Chapter 3 we focus on an alternative continuous-time formulation, where decisions can be taken at any point in time. We now describe both formulations of the service model in more detail.

1.3.1 The discrete-time service model

Consider a service center that is contracted for service to its customers within a prespecified time interval after the call for service has arrived. All customers have the same deterministic delay-limit of D periods, defined as the maximally allowable timespan between order arrival and service commencement. As long as the delay-limit of a customer has not yet expired, this customer can be accommodated together with other customers in one and the same batch. The cost associated with a batch service is a linear function of the number of customers included: the batch-service cost for a batch of size i is $a_B + b_B i$. Because of the fixed cost the average batch service cost per customer decreases with the batch size, i.e., economies of scale are are obtained for every additional customer. However, as soon as the delay-limit of a customer expires a more expensive individual service is required. The cost associated with an individual service is b_I (per customer). The service capacity of the system is assumed to be sufficiently large to accommodate any batch irrespective of its size and irrespective of other ongoing services. At equidistant points in time a decision has to be made whether or not to initiate a batch service. An optimal decision (in minimizing service costs) will not only depend on the total number of customers present at a decision epoch, but also on the distribution of the residual delays of those customers. This leads to a multi-dimensional state space, where the dimension corresponds to the number of time periods in the delay-limit; specifically, the state space becomes

$$\Omega_{\mathrm{DT}} := \{(r_0, \ldots, r_{D-1}) \mid r_i \in I\!N,\ i = 0, \ldots, D-1\}, \tag{1.1}$$

with r_i denoting the number of customers with a residual delay-limit of i periods, or the number of customers that require service within i periods (DT stands for "discrete time"). Using (1.1) a Markov decision process (see section 1.5) can be constructed to find an average-cost optimal policy, but due to the curse of dimensionality this is computationally feasible for moderate values of D only. Therefore we will consider several classes of restricted policies, based on a subset of the state information and characterized by a simple structure. These policies include:

- The Critical-Group policy: do a batch service if $r_0 \geq K$.

  Under this policy the batching decision is only based on the number of "urgent" customers (r_0), i.e., the customers whose delay-limit has expired and who require individual service when it is decided not to do a batch service. Notice that under the Critical-Group policy a batch service is never started within D periods since the previous batch service.

- The Total-Demand policy: do a batch service if $\sum\limits_{i=0}^{D-1} r_i \geq K$.
  Under this policy the batching decision is only based on the total number of waiting customers ($\sum_{i=0}^{D-1} r_i$). To prevent that a batch service is started within D periods since the previous batch service (which is suboptimal), this condition is imposed a priori.

- The Extended Total-Demand policy: do a batch service if $\sum\limits_{i=0}^{D-1} r_i \geq K_1$ and $r_0 \geq K_2$.
  This two-parameter policy is a combination of the other two policies: it bases the batching decision on the total number of customers as well as the number of urgent customers.

We complete the description of the discrete-time service model with a number of important observations that should illustrate the scope of the model (these observations also apply to the continuous-time service model).

i) The assumption of ample service capacity, together with the fact that the delay-limit does not include the service time, implies that the service time is completely irrelevant for the model. This distinguishes our model from a queueing model, where the service time is an important model characteristic (see also section 1.4).

ii) In situations where the delay-limit should include the service time (e.g., in the delivery example), the service model can still be applied provided that the delay-limit can be "adjusted" for the service (transportation) time. This is possible whenever no capacity constraints have to be imposed and the batch- and individual service time are both constant or bounded from above by a constant (say $L_{\max}$) that is smaller than D; it is easily verified that in this case an adjusted delay-limit $D - L_{\max}$ can be used. The case where the delay-limit includes a random and unbounded service time is incompatible with the rigid delay-limit constraint, and hence falls outside the scope of this thesis.

iii) It is intuitively clear that one should never start a batch service when no customer has reached the delay-limit (i.e., when $r_0 = 0$). In such a situation one better waits for another time period so that possibly more customers can be accommodated in the same batch. As a direct consequence, the time between two consecutive batch services will be at least D periods, and hence it suffices to assume that a batch service can always be carried out D periods after the previous one.

iv) If the individual service cost (b_I) is only slightly higher than the variable batch service cost (b_B), then the policy of providing all customers with an individual service is quite satisfactory and may even be optimal (in terms of average service costs per unit of time). On the other hand, if b_I is much higher than b_B and/or if the fixed batch-service cost (a_B) is relatively low, the policy of serving all customers through a batch service is likely to be (nearly) optimal.

1.3.2 The continuous-time service model

In the continuous-time version of the service model a batch or individual service can be started at any moment in time, and the delay-limit equals D time units. Furthermore, we assume that customer arrivals are governed by a Poisson process. Whereas in the discrete-time service model the waiting customers can be grouped into D "delay classes", a complete state description for the continuous-time model must include the residual delay-limit of every individual waiting customer. This would lead to a state space

$$\Omega_{\mathrm{CT}} := \{(n, d_1, \ldots, d_n) \mid n \in I\!N;\ 0 < d_1 < \cdots < d_n \leq D\}, \tag{1.2}$$

with n the number of waiting customers en d_i the residual delay-limit of the i^{th} customer (CT stands for "continuous time"). Since Ω_{CT} has infinite dimension, it is clearly impossible to compute an average-cost optimal policy (and if it were possible it would be of no practical use whatsoever). This forces us to restrict attention to heuristic policies such as the Total-Demand policy.

The key observation for the continuous-time analogon of the Total-Demand policy is the following. Suppose that $X(t)$ denotes the total number of waiting customers t time units after the last batch service, then $X(t)$ is equivalent to the number of customers in a M/D/∞ queue with constant service times D. This easily follows from the fact that, barring any intermediate batch services, any customer is given individual service upon expiration of his delay-limit (exactly D time units after arrival), so that

$$X(t) = \begin{cases} N(t) & \text{if } 0 \le t \le D; \\ N(t) - N(t-D) & \text{if } t > D. \end{cases} \tag{1.3}$$

Under the continuous-time Total-Demand policy with parameter K a batch service is started as soon as $X(t) \ge K$. The continuous-time analogon of the Critical-Group policy is not immediately clear. However, defining the critical group as the customers with a residual delay-limit of at most C time units (i.e., the customers that must be served within C time units), we are led to the broad class of "Generalized Critical-Group" policies: start a batch service if the number of customers with a residual delay-limit of C time units or less is at least K. Notice that for $C = D$ this is just the Total-Demand policy. The obvious decision problem is to find optimal values for the control parameters C and K, and it turns out that this involves finding the first entrance time into state K of a M/D/∞ queue with constant service times of C time units. This problem, that turns out to be remarkably difficult, is the subject of section 3.3 (see also [Jansen 1996]).

1.4 Related literature

The service model is perhaps best categorized as a stochastic clearing system with impatient customers, where clearing the system corresponds to a batch service and an impatient customer leaving the system to an individual service. Stochastic clearing systems are characterized by a non-decreasing stochastic input process $\{X(t), t \ge 0\}$, with $X(t)$ the total number of customers in the system at time t, and a clearing mechanism that instantaneously removes all present customers from the system (see [Stidham 1974] and [Stidham 1977]). The service model deviates from this set-up in that the input process is not non-decreasing, since customers whose delay-limit expires receive an individual service and thereby leave the system before it is "cleared" by a batch service ("impatient customers"); see (1.3).

Control of batch service queues is investigated in [Deb&Serfozo 1973], where the batch service times are i.i.d. random variables that are independent of the Poisson arrival process as well as the size of the batch. Besides a fixed batch service cost a waiting cost per customer per unit of time is assumed, and for this cost structure it is proved that the optimal policy is of the control-limit type: do a batch service whenever the number of customers in the system exceeds a certain threshold; see also [Weiss 1979]. Moreover, see [Ignall&Kolesar 1974], [Deb 1978], [Weiss 1981] and [Lee et al. 1994] for a related class of problems regarding the optimal dispatching of a passenger shuttle at a terminal, and [Avramidis&Uzsoy 1993] for a problem stemming from the area of production planning. Queueing models with impatient customers were introduced by [Palm 1937] and later investigated by many other authors, e.g., [Baccelli et al. 1984], [Stanford 1990] and [Boxma&de Waal 1994]. Finally, it is also worthwhile to mention the relationship between the service model and lotsizing models (see also Chapter 4). A related paper in that regard is [Dellaert&Melo 1995], where production to order with due dates is considered. Their model assumes a holding cost for orders finished before the due date as well as backordering against a penalty cost per period (note that individual services would correspond to lost sales). Using an N-dimensional Markov model (with N the maximal due date), heuristic dynamic lotsizing policies are analysed that aim at minimizing the total cost of holding, production and penalties. For a survey of other models concerned with batching decisions and lotsizing (deterministic as well as stochastic) we refer to [Kuik et al. 1994].

1.5 Preliminaries: Markov decision theory

In this section we briefly introduce the most important concepts from Markov decision theory, as they will be used throughout this thesis. For a thorough introduction to Markov decision theory we refer to [Tijms 1994] (Chapter 3), [Ross 1983], [Puterman 1994] and [Heyman&Sobel 1984] (Chapters 4 and 5).

Markov decision theory is concerned with the analysis of (semi-)Markov decision processes. Generally speaking, a Markov decision process (MDP) is a controlled Markov chain with a cost (or reward) structure, and a semi-Markov decision process (SMDP) is a controlled *embedded* Markov chain with a cost (or reward) structure. The primary goal of a SMDP formulation is to find optimal controls, i.e., controls that minimize a given cost criterion (or maximize a given reward criterion). Throughout this thesis we will only consider the criterion of expected long-run average costs per unit of time. Since a MDP is a special case of a SMDP, we only describe the SMDP framework here.

Consider a dynamic system that is reviewed at decision epochs $n = 1, 2, \ldots$ and define

$$\begin{aligned}
X_n &:= \text{state of the system at the } n^{\text{th}} \text{ decision epoch} \quad (n = 1, 2, \ldots); \\
\Omega &:= \text{set of all possible states of the system at any decision epoch;} \\
T_n &:= \text{time between the } (n-1)^{\text{th}} \text{ and } n^{\text{th}} \text{ decision epoch} \quad (n = 1, 2, \ldots); \\
C_n &:= \text{costs incurred between the } (n-1)^{\text{th}} \text{ and } n^{\text{th}} \text{ decision epoch} \quad (n = 1, 2, \ldots).
\end{aligned}$$

At each decision epoch, one out of a number of possible actions must be chosen. A policy $\mathbf{R}$ is a rule that prescribes which action to choose at any given decision epoch. We restrict attention to stationary policies, i.e., the action to be chosen at the n^{th} decision epoch only depends on the state of the system X_n and is independent of n and the past history of the process $\{X_n\}$. We define

$$\begin{aligned}
R_i &:= \text{action chosen in state } i \text{ under policy } \mathbf{R} \quad (i \in \Omega); \\
A(i) &:= \text{set of all possible actions when the state at a decision epoch is } i \quad (i \in \Omega).
\end{aligned}$$

For this controlled dynamic system to be a SMDP the following Markovian property must be satisfied: T_{n+1}, C_{n+1} and X_{n+1} only depend on $\{X_n, R_{X_n}\}$ and are independent of $\{X_1, R_{X_1}, \ldots, X_{n-1}, R_{X_{n-1}}\}$. Hence the one-step transition probabilities, transition times and transition costs of a SMDP are defined as

$$\begin{aligned}
p_{ij}(a) &:= \Pr\{X_{n+1} = j \mid X_n = i, R_i = a\} \quad (i, j \in \Omega;\ a \in A(i)); && (1.4) \\
\tau_i(a) &:= E\{T_{n+1} \mid X_n = i, R_i = a\} \quad (i \in \Omega;\ a \in A(i)); && (1.5) \\
c_i(a) &:= E\{C_{n+1} \mid X_n = i, R_i = a\} \quad (i \in \Omega;\ a \in A(i)), && (1.6)
\end{aligned}$$

respectively. In words: if at a given decision epoch action a is chosen in state i, then

i) the expected time until the next decision epoch is $\tau_i(a)$;

ii) the expected costs incurred until the next decision epoch are $c_i(a)$;

iii) the state at the next decision epoch is j with probability $p_{ij}(a)$.

Notice that only the *expected* transition times and the *expected* transition costs are needed; the underlying probability distributions are irrelevant. This makes the SMDP framework a very flexible modelling tool. A discrete-time Markov decision process (MDP) is just a SMDP with $\tau_i(a) = 1$ for all $i \in \Omega$ and $a \in A(i)$.

For a fixed stationary policy $\mathbf{R}$ a SMDP reduces to an embedded Markov chain on decision epochs $\{X_n(\mathbf{R})\}$ with

$X_n(\mathbf{R}) :=$ state of the system at the n^{th} decision epoch under policy $\mathbf{R}$ $(n = 1, 2, \ldots)$.

The one-step transition probabilities, transition times and transition costs of $\{X_n(\mathbf{R})\}$ are given by $p_{ij}(R_i)$ $(i, j \in \Omega)$, $\tau_i(R_i)$ $(i \in \Omega)$ and $c_i(R_i)$ $(i \in \Omega)$, respectively. Now suppose that $\{X_n(\mathbf{R})\}$ is aperiodic and irreducible, and define the n-step transition probabilities

$$p_{ij}^{(n)}(\mathbf{R}) := \Pr\{X_n(\mathbf{R}) = j \mid X_0(\mathbf{R}) = i\} \quad (i, j \in \Omega;\ n = 1, 2, \ldots), \tag{1.7}$$

then the stationary probabilities

$$\pi_j(\mathbf{R}) := \lim_{n\to\infty} p_{ij}^{(n)}(\mathbf{R}) \quad (j \in \Omega) \tag{1.8}$$

are the unique solution of

$$\pi_j(\mathbf{R}) = \sum_{i\in\Omega} \pi_i(\mathbf{R}) p_{ij}(R_i) \quad (j \in \Omega) \tag{1.9}$$

subject to $\sum_{j\in\Omega} \pi_j(\mathbf{R}) = 1$. Moreover, the expected average costs of policy $\mathbf{R}$ are given by

$$g(\mathbf{R}) := \lim_{n\to\infty} \frac{\sum_{m=0}^{n-1} \sum_{j\in\Omega} p_{ij}^{(m)}(\mathbf{R}) c_j(R_j)}{\sum_{m=0}^{n-1} \sum_{j\in\Omega} p_{ij}^{(m)}(\mathbf{R}) \tau_j(R_j)} = \frac{\sum_{i\in\Omega} \pi_i(\mathbf{R}) c_i(R_i)}{\sum_{i\in\Omega} \pi_i(\mathbf{R}) \tau_i(R_i)} \tag{1.10}$$

(see e.g. [Tijms 1994], Theorem 3.5.1).

A stationary policy $\mathbf{R}^*$ is *average-cost optimal* if

$$g := g(\mathbf{R}^*) = \min_{\mathbf{R}} g(\mathbf{R}). \tag{1.11}$$

For the exact conditions that guarantee existence of an average-cost optimal policy we refer to [Puterman 1994] (Chapter 8 and section 11.4). For most applications these conditions are not restrictive; in particular, these conditions are satisfied for all SMDP's that appear in this thesis (unless explicitly stated otherwise). If an optimal policy exists, then there exist *relative values* v_i $(i \in \Omega)$ such that g and v_i $(i \in \Omega)$ satisfy the *average-cost optimality equations*

$$v_i = \min_{a \in A(i)} \Big\{ c_i(a) - g\tau_i(a) + \sum_{j \in \Omega} p_{ij}(a) v_j \Big\} \quad (i \in \Omega) \tag{1.12}$$

(see e.g. [Tijms 1994], p. 221). The v_i $(i \in \Omega)$ are not uniquely determined, i.e., if $\{g;\ v_i, i \in \Omega\}$ is a solution of (1.12) then so is $\{g;\ v_i + c, i \in \Omega\}$ for any constant c. A unique solution is obtained by setting $v_k = 0$ for any $k \in \Omega$. The difference $v_i - v_j$ can be interpreted as the additional total costs incurred over an infinite horizon when starting in state i instead of state j and following an optimal policy. As a result, the v_i $(i \in \Omega)$ provide an ordering of all states in Ω: state i is "better" than state j if $v_i < v_j$.

Mainly because of the non-linearity, the optimality equations (1.12) are not easily solved and an iterative method is needed to find an optimal policy $\mathbf{R}^*$. The most widely used solution methods include policy iteration and value iteration, and we now describe these two algorithms.

Policy iteration

The policy-iteration algorithm starts with an arbitrary policy $\mathbf{R}_0$, and constructs an improved policy in every step until an optimal policy is found and no further improvement is possible. It consists of the following steps:

0 *initialization* $\mathbf{R} := \mathbf{R}_0$.

1 *value determination* Compute $g(\mathbf{R})$ and $v_i(\mathbf{R})$ $(i \in \Omega)$ by solving

$$\begin{aligned} v_k(\mathbf{R}) &= 0; \\ v_i(\mathbf{R}) &= c_i(R_i) - g(\mathbf{R})\tau_i(R_i) + \sum_{j \in \Omega} p_{ij}(R_i) v_j(\mathbf{R}) \quad (i \in \Omega), \end{aligned} \tag{1.13}$$

with $k \in \Omega$ arbitrary.

2 *policy improvement* Compute an improved policy $\mathbf{R}'$ from

$$R'_i = \arg \min_{a \in A(i)} \Big\{ c_i(a) - g(\mathbf{R})\tau_i(a) + \sum_{j \in \Omega} p_{ij}(a) v_j(\mathbf{R}) \Big\} \quad (i \in \Omega). \tag{1.14}$$

3 *convergence test* If $\mathbf{R}' = \mathbf{R}$ then stop; else $\mathbf{R} := \mathbf{R}'$; go to 1.

Value iteration

The value-iteration algorithm (or "successive approximations") uses dynamic programming to approximate the infinite-horizon problem by a sequence of finite-horizon problems. Since dynamic programming can only be used for discrete-time problems, the SMDP is first transformed to a discrete-time MDP by using the data transformation

$$
\begin{aligned}
\tau &:= \min_{i\in\Omega, a\in A(i)} \tau_i(a); \\
\tau_i'(a) &:= 1 \quad (i \in \Omega;\ a \in A(i)); \\
c_i'(a) &:= \frac{c_i(a)}{\tau_i(a)} \quad (i \in \Omega;\ a \in A(i)); \\
p_{ij}'(a) &:= \begin{cases} \frac{\tau}{\tau_i(a)} p_{ij}(a) & \text{if } j \neq i \\ \frac{\tau}{\tau_i(a)} p_{ij}(a) + 1 - \frac{\tau}{\tau_i(a)} & \text{if } j = i \end{cases} \quad (i, j \in \Omega;\ a \in A(i)).
\end{aligned}
\tag{1.15}
$$

The value-iteration algorithm consists of the following steps:

0 *initialization* $v_0(i) := 0$ $(i \in \Omega)$; $n := 1$.

1 *induction step* Compute $\{v_n(i), i \in \Omega\}$ from

$$v_n(i) = \min_{a\in A(i)} \Big\{ c_i'(a) + \sum_{j\in\Omega} p_{ij}'(a) v_{n-1}(j) \Big\} \quad (i \in \Omega). \tag{1.16}$$

2 *convergence test* Compute

$$w_n := \min_{i\in\Omega}\{v_n(i) - v_{n-1}(i)\}, \quad W_n := \max_{i\in\Omega}\{v_n(i) - v_{n-1}(i)\}. \tag{1.17}$$

If $\frac{W_n - w_n}{w_n} < \epsilon$ then stop; else $n := n + 1$; go to 1.

In general, policy iteration is more efficient than value iteration in terms of computation time, because the number of iterations is usually much smaller; the numerical results to follow will confirm this. Also the policy iteration algorithm computes the exact value of g, while the value iteration algorithm only approximates g with an accuracy of ϵ (but the exact value of g can easily be computed afterwards by solving (1.13) with $\mathbf{R}$ the optimal policy).

Part I

The service model

Chapter 2

The discrete-time service model

2.1 Introduction

In this chapter we deal with the discrete-time service model introduced in Chapter 1 (see also [Berg et al. 1996] and [Berg et al. 1998]). Decisions can only be made at equidistant points in time $t, 2t, \ldots$, where t is usually determined on the basis of operational considerations and thus varies among applications (e.g., every day or every week). Thus, we define period n as the time slot $((n-1)t, nt]$ and the n^{th} decision epoch as the end of period n ($n = 1, 2, \ldots$). This allows us to treat all customers arriving during a period as if they arrived together at the beginning of that period. Moreover, we may assume a general discrete pdf for the distribution of the number of customers arriving in period n. This is an important advantage of the discrete-time model over the continuous-time model, where we restrict ourselves to a Poisson arrival process.

Let X_n denote the number of customers who arrive during period n ($n = 1, 2, \ldots$). The X_n are assumed to be i.i.d. random variables with probability distribution function (pdf) $q_k := \Pr\{X_n = k\}$, cumulative distribution function (cdf) $Q_k := \sum_{j=0}^{k} q_j$ ($k = 0, 1, \ldots$) and mean μ. In the special case of a Poisson arrival process with rate λ, X_n has a Poisson(λt) distribution with $q_k = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$ and $\mu = \lambda t$. The delay-limit is defined as the maximally allowable timespan between the arrival of a customer and the start of his service, i.e., service for any customer must commence within the delay-limit. We assume that the delay-limit (maximal waiting time, "impatience") is constant and equal to D periods. In order to satisfy the rigid delay-limit constraint, there are two possible service modes: batch service (for a group of customers) and individual service. At any decision epoch a batch service can be started that accommodates all waiting customers (the system is "cleared"); we assume that there is ample batch service capacity. The cost associated with a batch of i customers is $a_B + b_B i$, where the fixed cost a_B is usually much larger than the variable cost b_B. Because of the economies of scale there is an incentive to include as many customers as possible in a batch. When the delay-limit of a customer expires (i.e., the customer is waiting for service for D periods), and it is decided not to start a batch service, an individual service is mandatory at cost b_I. An individual service is relatively costly; we assume w.l.o.g. that $a_B > b_I > b_B$ (for $b_I \leq b_B$ the optimal policy is to provide all customers with an individual service upon expiration of the delay-limit). Note that we assume no fixed individual service cost, so that no economies of scale can be obtained from grouping individual services (this explains the term "individual service"). We also assume that there are no restrictions on the number of simultaneous individual services.

The problem is to determine at which decision epochs a batch service should be started in order to minimize the long run expected total service costs per period subject to the delay-limit constraint. In other words, the objective is to find an optimal trade-off between batch and individual services. It follows from the assumption of ample service capacity that neither the service time distribution nor the mean service time is relevant for this problem. Also observe that, due to the absence of waiting costs, a customer will not receive individual service as long as his delay-limit has not yet expired.

Throughout this chapter we will focus on the case $D > 1$, since for $D = 1$ the optimal policy is trivial: at the n^{th} decision epoch start a batch service if $a_B + b_B X_n \leq b_I X_n$, or $X_n \geq \lceil \frac{a_B}{b_I - b_B} \rceil$. Conditioning on X_n we see that the expected average costs for this policy are given by

$$\sum_{k=0}^{\lceil \frac{a_B}{b_I-b_B}\rceil -1} q_k\, b_I k + \sum_{k=\lceil \frac{a_B}{b_I-b_B}\rceil}^{\infty} q_k(a_B + b_B k) = b_B \mu + b_I \sum_{k=0}^{\lceil \frac{a_B}{b_I-b_B}\rceil -1} k q_k + a_B\Big(1 - Q_{\lceil \frac{a_B}{b_I-b_B}\rceil -1}\Big). \quad (2.1)$$

For $D > 1$ a complete state description, including all relevant information about the state of the system at decision epochs, is provided by the D-dimensional vector

$$\mathbf{r} := (r_0, \ldots, r_{D-1}) \quad (r_i \in I\!N,\ i = 0, \ldots, D-1), \quad (2.2)$$

where r_i denotes the number of waiting customers with a residual delay-limit of i periods (i.e., the number of waiting customers that must be served within i periods). In any state $\mathbf{r}$ there are two possibilities: either serve all $\sum_{i=0}^{D-1} r_i$ customers through a batch service, or serve r_0 customers individually. A stationary policy π is a function from $I\!N^D$ to $\{0, 1\}$, specifying for any state $\mathbf{r}$ whether to start a batch service (1) or not (0). Every such policy π generates a stochastic process $\{\mathbf{R}^{(n)},\ n = 1, 2, \ldots\}$, where $\mathbf{R}^{(n)} := (R_0^{(n)}, \ldots, R_{D-1}^{(n)})$ denotes the state vector at the n^{th} decision epoch (the dependence on π is omitted for ease of notation). By virtue of the i.i.d. arrival process $\{X_n\}$ and the stationary policy π, $\{\mathbf{R}^{(n)}\}$ is a (discrete-time and discrete-state) Markov chain with state space $I\!N^D$. This enables us to construct a Markov decision process (MDP) with which the average-cost optimal policy π^* can be computed, and we will study the MDP formulation in section 2.6. It turns out that the optimal policy does not have a simple structure, making it of little practical use. Moreover, due to the curse of dimensionality, computing the optimal policy numerically is only feasible for small values of D (say $D \leq 3$). Therefore it is important to look for restricted policies with a simple structure that are easier to compute and still close to the global optimal policy (when optimized with regard to their respective policy parameters).

The rest of this chapter is organized as follows. In section 2.2 we derive a general expression for the expected average costs of a stationary policy, using the fact that any given policy induces a regenerative process with batch-service epochs as regeneration epochs. We also introduce two extreme policies: under the Never-Batch policy all customers are served individually, while under the Only-Batch policy all customers are served through a batch service. In section 2.3 we analyse the class of Critical-Group policies, which are exclusively based on the number of customers whose delay-limit is about to expire (r_0). In section 2.4 the class of Total-Demand policies, based on the total number of customers in the system ($\sum_{i=0}^{D-1} r_i$), is investigated. In section 2.5 we describe two more sophisticated policies: the three-parameter Extended Critical-Group policy that adds extra flexibility to the Critical-Group policy, and the two-parameter Extended Total-Demand policy that combines the elements of the Critical-Group and the Total-Demand policy. In section 2.6 we turn to the computation of the optimal policy by means of a Markov decision process, and derive some structural properties of the optimal policy. In section 2.7 numerical experiments are reported that compare the performance of the various heuristic policies with the global optimal policy, thereby establishing the value of detailed information on the customer's residual delay-limits. Most of the numerical experiments are limited to small values of D, but in section 2.8 we argue that this is not really restrictive; the parameter D can also be seen as a decision variable determining the frequency of decision epochs.

2.2 Preliminary results

First we derive a general expression for the expected average costs per period of an arbitrary stationary policy π. Clearly, under every policy π the resulting vector-valued stochastic process $\{\mathbf{R}^{(n)},\ n = 1, 2, \ldots\}$ is regenerative, with regeneration epochs the decision epochs at which the system is cleared by a batch service. Therefore the analysis of all classes of special structured policies can be based on the theory of regenerative stochastic processes. For a given policy π, let a (regenerative) cycle be the time between two consecutive batch services and define

S_π := cycle length in periods;
Y_π := number of individual services during a cycle;
Z_π := number of customers included in the batch service at the end of a cycle;
N_π := total number of customers in a cycle;
g_π := expected average costs per period under policy π.

By the construction of the model we have the following relations:

$$Y_\pi = \sum_{n=1}^{S_\pi - D} X_n, \quad Z_\pi = \sum_{n=S_\pi-D+1}^{S_\pi} X_n, \quad N_\pi = \sum_{n=1}^{S_\pi} X_n = Y_\pi + Z_\pi \tag{2.3}$$

(where the numbering of X_n starts anew at the beginning of every cycle). It can be shown that for any policy π that prescribes a batch service in at least one state $\mathbf{r}$, S_π is stochastically dominated by a geometrically distributed random variable. Hence the only policy for which $E\{S_\pi\} = \infty$ is the policy that never prescribes a batch service. However, the expected average costs of this extreme policy, denoted by NB (Never-Batch), are simply given by

$$g_{\text{NB}} = \sum_{k=0}^{\infty} q_k\, b_I k = b_I \mu. \tag{2.4}$$

For any other policy we have by the Renewal Reward Theorem (see e.g. [Tijms 1994], Theorem 1.3.1) that the expected average costs per period are given by

$$g_\pi = \frac{a_B + b_B E\{Z_\pi\} + b_I E\{Y_\pi\}}{E\{S_\pi\}}. \tag{2.5}$$

The event $\{S_\pi = n\}$ is completely determined by $X_1, \ldots, X_n$, since the batch-service decision is completely determined by $\mathbf{R}^{(n)}$ which in turn is completely determined by $X_1, \ldots, X_n$. Hence S_π is a stopping time for $\{X_n\}$, and we have, by applying Wald's equation, that

$$E\{N_\pi\} = E\Big\{\sum_{n=1}^{S_\pi} X_n\Big\} = \mu E\{S_\pi\}. \tag{2.6}$$

Combining (2.3), (2.5) and (2.6) we obtain

$$g_\pi = \frac{a_B + b_B\big(\mu E\{S_\pi\} - E\{Y_\pi\}\big) + b_I E\{Y_\pi\}}{E\{S_\pi\}} = b_B\mu + \frac{a_B + (b_I - b_B)E\{Y_\pi\}}{E\{S_\pi\}}. \tag{2.7}$$

It follows from (2.7) that π^* depends on a_B, b_B and b_I only through $\frac{a_B}{b_I - b_B}$. Therefore we may assume w.l.o.g. in the remainder of the chapter that $b_B = 0$ and $b_I = 1$, which leaves us with only one (standardized) cost parameter a_B. It is also intuitively clear that $S_{\pi^*} \geq D$, as no individual services are required during the first D periods of a cycle (we give a formal proof of this proposition in section 2.6).

Under the NB-policy all customers are served individually; the other extreme is a policy where all customers are served through a batch service. The most obvious policy of this type is a periodic policy where a batch service is done every D periods, with average costs of a_B/D. However, this policy can be improved by postponing the batch service when no individual services are needed. As will become clear in the next section, this policy corresponds to a Critical-Group policy with control-limit 1. Denoting this policy by OB (Only-Batch), we have by (2.19) with $K = 1$ that

$$g_{\text{OB}} = \frac{a_B(1 - q_0)}{D(1 - q_0) + q_0} \tag{2.8}$$

(this can also be verified independently). The NB- and OB-policy can be regarded as two extreme policies, in the sense that π^* converges to the NB-policy if a_B tends to ∞ and to the OB-policy if a_B tends to 0.

2.3 The Critical-Group policy

In this section we consider the class of policies that are exclusively based on r_0, the number of customers whose delay-limit is about to expire and who require individual service if it is decided not to do a batch service. We will call these policies Critical-Group policies and denote them by CG. The following theorem confirms the obvious fact that it suffices to consider so-called "control-limit" policies.

Theorem 2.1 *The average-cost optimal policy within the class of Critical-Group policies is of control-limit type, i.e., start a batch service at the* n^{th} *decision epoch if and only if* $R_0^{(n)} \geq K$ *for some threshold value* $K \geq 1$.

Proof. See Appendix 2.A.

In order to compute the expected average costs for a CG-policy with control-limit K, we derive expressions for $E\{Y_{\text{CG}}\}$ and $E\{S_{\text{CG}}\}$, and then apply (2.7) (we omit the dependence on K for ease of notation).

Define

$$T := \min\{n = 1, 2, \ldots : X_n \geq K\} \tag{2.9}$$

as the first period in which K or more customers arrive. A batch service will be started as soon as the delay-limit of this "critical group" expires, implying that

$$S_{\text{CG}} = \min\{n = 1, 2, \ldots : R_0^{(n)} \geq K\} = T + D - 1. \tag{2.10}$$

Since T is geometrically distributed with parameter $1 - Q_{K-1}$, it follows that

$$\Pr\{S_{\text{CG}} = n\} = Q_{K-1}^{n-D}(1 - Q_{K-1}) \quad (n \geq D), \tag{2.11}$$

and hence

$$E\{S_{\text{CG}}\} = E\{T\} + D - 1 = D - 1 + \frac{1}{1 - Q_{K-1}} = D + \frac{Q_{K-1}}{1 - Q_{K-1}}. \tag{2.12}$$

Given $\{S_{\text{CG}} = n\}$ and $n > D$, the number of individual services in a cycle is equal to the number of customers that arrived in periods $1, \ldots, n - D$, or

$$E\{Y_{\text{CG}} \mid S_{\text{CG}} = n\} = \sum_{i=1}^{n-D} E\{X_i \mid S_{\text{CG}} = n\} \quad (n > D). \tag{2.13}$$

By the independence of the X_n $(n = 1, 2, \ldots)$,

$$\begin{aligned} E\{X_i \mid S_{\text{CG}} = n\} &= E\{X_i \mid X_1 < K, \ldots, X_{n-D} < K, X_{n-D+1} \geq K\} \\ &= E\{X_i \mid X_i < K\} = \frac{\sum_{k=0}^{K-1} k q_k}{Q_{K-1}} \quad (n > D;\ i = 1, \ldots, n - D). \end{aligned} \tag{2.14}$$

From (2.13) and (2.14) we conclude that

$$E\{Y_{\text{CG}} \mid S_{\text{CG}} = n\} = (n - D)\frac{\sum_{k=0}^{K-1} k q_k}{Q_{K-1}} \quad (n > D), \tag{2.15}$$

which implies with (2.11) that

$$E\{Y_{\text{CG}}\} = \sum_{n=D}^{\infty} Q_{K-1}^{n-D}(1 - Q_{K-1})(n - D)\frac{\sum_{k=0}^{K-1} k q_k}{Q_{K-1}} = \frac{\sum_{k=0}^{K-1} k q_k}{1 - Q_{K-1}}. \tag{2.16}$$

An alternative derivation of $E\{Y_{\text{CG}}\}$ is the following. Using (2.3) and Wald's equation we have that

$$E\{Y_{\text{CG}}\} = E\Big\{\sum_{n=1}^{T-1} X_n\Big\} = E\Big\{\sum_{n=1}^{T} X_n - X_T\Big\} = E\{T\}E\{X_1\} - E\{X_T\}, \tag{2.17}$$

since T is a stopping time for $\{X_n\}$ while $T-1$ is not. Now

$$\begin{aligned} E\{X_T\} &= \sum_{n=1}^{\infty} \Pr\{T=n\}E\{X_n \mid T=n\} = \sum_{n=1}^{\infty} \Pr\{T=n\}E\{X_n \mid X_n \geq K\} \\ &= E\{X_1 \mid X_1 \geq K\} = \frac{\sum_{k=K}^{\infty} kq_k}{1-Q_{K-1}}, \end{aligned} \tag{2.18}$$

and hence

$$E\{Y_{\text{CG}}\} = \frac{\sum_{k=0}^{\infty} kq_k}{1-Q_{K-1}} - \frac{\sum_{k=K}^{\infty} kq_k}{1-Q_{K-1}} = \frac{\sum_{k=0}^{K-1} kq_k}{1-Q_{K-1}},$$

in accordance with (2.16).

Substituting (2.12) and (2.16) into (2.7) (with $b_B = 0$ and $b_I = 1$) we obtain the following expression for the expected average costs per period in terms of the control-limit K:

$$g_{\text{CG}}(K) = \frac{a_B + \frac{1}{1-Q_{K-1}}\sum_{k=0}^{K-1} kq_k}{D + \frac{Q_{K-1}}{1-Q_{K-1}}} = \frac{a_B(1-Q_{K-1}) + \sum_{k=0}^{K-1} kq_k}{D(1-Q_{K-1}) + Q_{K-1}}. \tag{2.19}$$

Theorem 2.2 *Within the class of CG-policies the optimal control-limit K^* can be characterized as the smallest value of K satisfying the inequality*

$$K + (D-1)\sum_{k=0}^{K-1}(1-Q_k) \geq a_B.$$

Moreover, $K^* \leq \lceil a_B \rceil$.

Proof. From (2.19) it follows that

$$\begin{aligned} & g_{\text{CG}}(K+1) \geq g_{\text{CG}}(K) \\ \Leftrightarrow\quad & \left(a_B(1-Q_K) + \sum_{k=0}^{K} kq_k\right)(D(1-Q_{K-1}) + Q_{K-1}) \geq \\ & \left(a_B(1-Q_{K-1}) + \sum_{k=0}^{K-1} kq_k\right)(D(1-Q_K) + Q_K) \\ \Leftrightarrow\quad & \frac{DKq_K - (D-1)\left(Q_{K-1}\sum_{k=0}^{K} kq_k - Q_K\sum_{k=0}^{K-1} kq_k\right)}{q_K} \geq a_B. \end{aligned} \tag{2.20}$$

Next, using the fact that

$$\sum_{k=0}^{K-1} kq_k = \sum_{k=1}^{K-1}\sum_{l=0}^{k-1} q_k = \sum_{l=0}^{K-2}\sum_{k=l+1}^{K-1} q_K = \sum_{l=0}^{K-2}(Q_{K-1} - Q_l) = (K-1)Q_{K-1} - \sum_{k=0}^{K-2} Q_k, \quad (2.21)$$

we have that

$$Q_{K-1}\sum_{k=0}^{K} kq_k - Q_K\sum_{k=0}^{K-1} kq_k = Q_{K-1}\sum_{k=0}^{K-1} kq_k + Q_{K-1}Kq_K - (Q_{K-1}+q_K)\sum_{k=0}^{K-1} kq_k =$$
$$Q_{K-1}Kq_K - q_K\sum_{k=0}^{K-1} kq_k = q_K\left(KQ_{K-1} - (K-1)Q_{K-1} + \sum_{k=0}^{K-2} Q_k\right) = q_K\sum_{k=0}^{K-1} Q_k. \quad (2.22)$$

Substituting (2.22) into (2.21) we obtain

$$\begin{aligned} g_{\mathrm{CG}}(K+1) \geq g_{\mathrm{CG}}(K) \;&\Leftrightarrow\; DK - (D-1)\sum_{k=0}^{K-1} Q_k \geq a_B \\ &\Leftrightarrow\; K + (D-1)\sum_{k=0}^{K-1}(1-Q_k) \geq a_B. \end{aligned} \quad (2.23)$$

Define

$$f(K) \;:=\; K + (D-1)\sum_{k=0}^{K-1}(1-Q_k), \quad (2.24)$$

then

$$f(K) - f(K-1) = 1 + (D-1)(1-Q_{K-1}) \geq 1, \quad (2.25)$$

so that $f(K)$ is an increasing function of K. Moreover, $f(0) = 0 < a_B$ and $\lim_{K\to\infty} f(K) = \infty$, whence $g_{\mathrm{CG}}(K)$ has a unique minimum characterized as the smallest value of K for which (2.23) holds. Finally, $f(\lceil a_B \rceil) \geq a_B$, implying that $K^* \leq \lceil a_B \rceil$. □

2.4 The Total-Demand policy

Another class of realistic policies consists of those policies based on the total number of waiting customers in the system. More specifically, define L_n as the total number of customers present at the n^{th} decision epoch under the NB-policy, $S_n := \sum_{i=1}^{n} X_i$ and $S_{m,n} := \sum_{i=m}^{n} X_i$ ($n = 1, 2, \ldots;\ m = 1, \ldots, n$), then clearly

$$L_n = \begin{cases} S_n & \text{if } n \leq D; \\ S_{n-D+1,n} & \text{if } n > D. \end{cases} \quad (2.26)$$

The simplest policy in this class is a policy π that prescribes a batch service at $S_\pi = \min\{n = 1, 2, \ldots : L_n \geq K\}$. However, we have already noted that it is suboptimal to start a batch service earlier than D periods since the previous batch service (by waiting for D periods more customers can be included in the same batch without the need for extra individual services). This observation is easily incorporated by imposing the restriction that $S_\pi \geq D$. Denote the resulting policy by TD (Total-Demand), then

$$S_{\mathrm{TD}} = \min\{n = D, D+1, \ldots : L_n \geq K\}. \tag{2.27}$$

Our aim is to find the control-limit K for which the expected average costs per period $g_{\mathrm{TD}}(K)$ are minimal. Again we use (2.7) and focus on $E\{S_{\mathrm{TD}}\}$ and $E\{Y_{\mathrm{TD}}\}$.

For $E\{S_{\mathrm{TD}}\}$ we have by (2.27) that

$$\begin{aligned} E\{S_{\mathrm{TD}}\} &= D + \sum_{n=D}^{\infty} \Pr\{S_{\mathrm{TD}} > n\} \\ &= D + \sum_{n=D}^{\infty} \Pr\{L_i < K,\ i = D, \ldots, n\} \\ &= D + \sum_{n=D}^{\infty} \Pr\{S_{i,i+D-1} < K,\ i = 1, \ldots, n - D + 1\}. \end{aligned} \tag{2.28}$$

It turns out that (2.28) cannot be simplified further. The main reason is that $\{L_n\}$ is not a Markov chain, so that $\Pr\{S_{\mathrm{TD}} > n\}$ $(n \geq D)$ cannot be computed recursively. Instead we develop a computational scheme for $E\{S_{\mathrm{TD}}\}$, using the functions

$$\begin{aligned} Q_n(k_1, \ldots, k_n) &:= \Pr\{S_i \leq k_i,\ i = 1, \ldots, n\} \quad (n = 1, \ldots, D-1); \\ Q_n(k_1, \ldots, k_{D-1}) &:= \Pr\{S_i \leq k_i,\ i = 1, \ldots, D{-}1; S_{i-D+1,i} < K,\ i = D, \ldots, n\}\ (n \geq D). \end{aligned}$$

Conditioning on X_n we obtain the following recursive relations for the function values of Q_n:

$$\begin{aligned} Q_1(k_1) &= Q_{k_1}; \\ Q_n(k_1, \ldots, k_n) &= \sum_{k=0}^{k_1} q_k Q_{n-1}(k_2 - k, \ldots, k_n - k) \quad (n = 2, \ldots, D-1); \\ Q_n(k_1, \ldots, k_{D-1}) &= \sum_{k=0}^{k_1} q_k Q_{n-1}(k_2 - k, \ldots, k_{D-1} - k, K - 1 - k) \quad (n \geq D). \end{aligned} \tag{2.29}$$

Define $P_n := \Pr\{S_{\mathrm{TD}} > n\}$ $(n \geq D)$ and $k_{m,n} := \sum_{i=m}^{n} k_i$ $(m \leq n)$, then by conditioning on $(X_1, \ldots, X_{D-1})$ we find that

$$P_n = \sum_{k_{1,D-1} < K} q_{k_1} \cdots q_{k_{D-1}} Q_{n-D+1}(K - 1 - k_{1,D-1}, \ldots, K - 1 - k_{n-D+1,D-1}) \tag{2.30}$$

for $n = D, \ldots, 2D-2$, and

$$P_n = \sum_{k_{1,D-1} < K} q_{k_1} \cdots q_{k_{D-1}} Q_{n-D+1}(K - 1 - k_{1,D-1}, \ldots, K - 1 - k_{D-1}) \tag{2.31}$$

for $n \geq 2D-1$. Finally, by (2.28) we have that

$$E\{S_{\mathrm{TD}}\} = D + \sum_{n=D}^{\infty} P_n, \tag{2.32}$$

and since $\lim_{n\to\infty} P_n = 0$ we truncate the infinite sum if P_n is sufficiently small.

Remark. Numerical analysis reveals that, asymptotically, P_n constitutes a geometric series, or

$$\lim_{n\to\infty} \frac{P_n}{P_{n-1}} = C \quad (0 < C < 1) \tag{2.33}$$

(we were not able to give a formal proof of (2.33), nor to find an explicit expression for the limit C). Based on (2.33) we replace the tail of the infinite sum by a geometric series, which leads to the sequence of approximations

$$D + \sum_{n=D}^{N} P_n + \frac{P_{N-1}P_N}{P_{N-1} - P_N} \quad (N = D+1, D+2, \ldots). \tag{2.34}$$

It turns out that (2.34) converges extremely fast to $E\{S_{\mathrm{TD}}\}$.

To compute $E\{Y_{\mathrm{TD}}\}$ we use the relation

$$E\{Y_\pi\} = E\{N_\pi\} - E\{Z_\pi\} = \mu E\{S_\pi\} - E\{Z_\pi\}, \tag{2.35}$$

that follows from (2.3) and (2.6). The idea is that for the TD-policy $E\{Z_\pi\}$ is easier to compute than $E\{Y_\pi\}$. We have that

$$\begin{aligned} E\{Z_\pi\} &= \sum_{n=D}^{\infty} \Pr\{S_\pi = n\} E\{S_{n-D+1,n} \mid S_\pi = n\} \\ &= \sum_{n=D}^{\infty} \Pr\{S_\pi = n\} \sum_{k=0}^{\infty} \Pr\{S_{n-D+1,n} > k \mid S_\pi = n\} \\ &= \sum_{n=D}^{\infty} \Pr\{S_\pi = n\} \left(K + \sum_{k=K}^{\infty} \Pr\{S_{n-D+1,n} > k \mid S_\pi = n\} \right) \\ &= K + \sum_{n=D}^{\infty} \sum_{k=K}^{\infty} \Pr\{S_{n-D+1,n} > k, S_\pi = n\}. \end{aligned} \tag{2.36}$$

We use (2.36) to set up a computational scheme for $E\{Y_{\mathrm{TD}}\}$. Define

$$y_n := \sum_{k=K}^{\infty} \Pr\{S_{n-D+1,n} > k, S_{\mathrm{TD}} = n\} \quad (n = D, D+1, \ldots). \tag{2.37}$$

For $n = D$ we have that

$$y_D = \sum_{k=K}^{\infty} \Pr\{S_D > k\} = E\{S_D\} - \sum_{k=0}^{K-1} \Pr\{S_D > k\} = D\mu - \sum_{k=0}^{K-1} \left(1 - Q_k^{(D)}\right), \tag{2.38}$$

where $Q_k^{(n)} := \Pr\{S_n \le k\}$ is the n-fold convolution of Q_k $(k = 0, 1, \ldots)$. For $n > D$ we have that

$$\begin{aligned}
y_n &= \sum_{k=K}^{\infty} \Pr\{S_{n-D+1,n} > k, S_{\mathrm{TD}} = n\} \\
&= \sum_{k=K}^{\infty} \Pr\{S_D < K, \ldots, S_{n-D,n-1} < K, S_{n-D+1,n} > k\} \\
&= \sum_{k=K}^{\infty} \sum_{k_{n-D+1,n-1}<K} q_{k_{n-D+1}} \cdots q_{k_{n-1}} \Pr\{X_n > k - k_{n-D+1,n}\} \cdot \\
&\Pr\{X_{n-D} < K - k_{n-D+1,n-1}, \ldots, S_{n-2D+2,n-D} < K - k_{n-D+1}, S_{n-2D+1,n-D} < K, \ldots, S_D < K\} \\
&= \sum_{k_{n-D+1,n-1}<K} q_{k_{n-D+1}} \cdots q_{k_{n-1}} \Big(\mu - \sum_{k=0}^{K-1-k_{n-D+1,n-1}} (1 - Q_k)\Big) \cdot \\
&\quad Q_{n-D+2}(K-1-k_{n-D+1,n-1}, \ldots, K-1-k_{n-D+1}),
\end{aligned} \tag{2.39}$$

where we compute $Q_n(\cdot)$ recursively as in (2.29). Note that for $n = D$ (2.39) does *not* reduce to (2.38). Combining (2.35), (2.36) and (2.37) we obtain

$$E\{Y_{\mathrm{TD}}\} = \mu E\{S_{\mathrm{TD}}\} - K - \sum_{n=D}^{\infty} y_n. \tag{2.40}$$

Since $\lim_{n\to\infty} y_n = 0$ we truncate the infinite sum if y_n is sufficiently small.

Although the computation time of the numerical schemes for $E\{S_{\mathrm{TD}}\}$ and $E\{Y_{\mathrm{TD}}\}$ increases exponentially with D and K, for $D \leq 5$ the computation time remains within the order of seconds. As an alternative to this probabilistic method, $E\{S_{\mathrm{TD}}\}$ and $E\{Y_{\mathrm{TD}}\}$ can also be computed by a "brute-force" method in which two $(D-1)$-dimensional systems of equations must be solved (see Appendix 2.B). For this method the computation time also increases exponentially, but considerably faster than for the probabilistic method.

2.5 Extended policies

In this section we introduce two more sophisticated policies that extend the CG-policy of section 2.3 and the TD-policy of section 2.4, respectively. These policies are made more flexible by adding one or two extra parameters, giving a better performance at the expense of increased model complexity.

2.5.1 The Extended Critical-Group policy

The basic idea of the CG-policy, ensuring that the (large) critical group is included in the batch, is still captured if we allow the batch service to be started at *any* of the next $D-1$ decision epochs $T+1, \ldots, T+D-1$, and not necessarily at $T+D-1$ as in (2.10). With this in mind we extend the CG-policy such that a batch service will be done at the first decision epoch after T at which the total number of waiting customers is large enough, i.e., exceeds some control-limit. However, the relevant quantity should only include the customers who arrived prior to T and are still waiting for service; the customers arriving after T will be included in the batch in any case and should not influence the batch-service decision. We also impose an additional condition requiring that the current critical group (r_0) is large enough, thereby adding a third control-limit. Specifically, under this so-called Extended Critical-Group (ECG) policy, a batch is started at $T_1 + T_2$ with

$$T_1 := \min\{n = 1, 2, \dots : X_n \geq K_1\}; \tag{2.41}$$

$$T_2 := \min\Big\{D-1, \min\Big\{n = 0, \dots, D-2 : \frac{\sum_{m=n}^{D-2} R_m^{(T_1)}}{D-n-1} \geq K_2 \text{ and } R_n^{(T_1)} \geq K_3\Big\}\Big\}. \tag{2.42}$$

In words: wait for a group of at least K_1 customers, and do a batch service as soon as the average number of individual services per period to be avoided is at least K_2 *and* the size of the current critical group is at least K_3, where the batch service must be done at most $D-1$ periods after the arrival of the critical group. Hence the ECG-policy uses three control parameters K_1, K_2 and K_3, of which K_2 is a continuous variable. The search for the optimal values K_1^*, K_2^* and K_3^* requires the computation of the expected average cost $g_{\text{ECG}}(K_1, K_2, K_3)$, which in turn requires the computation of $E\{S_{\text{ECG}}\}$ and $E\{Y_{\text{ECG}}\}$.

Define

$$U_1 := \sum_{n=1}^{T_1-D} X_n; \tag{2.43}$$

$$U_2 := \sum_{n=T_1-D+1}^{T_1+T_2-D} X_n = \sum_{n=0}^{T_2-1} R_n^{(T_1)}, \tag{2.44}$$

the number of individual services during T_1 and T_2, respectively. Clearly,

$$E\{S_{\text{ECG}}\} = E\{T_1\} + E\{T_2\}; \tag{2.45}$$

$$E\{Y_{\text{ECG}}\} = E\{U_1\} + E\{U_2\}, \tag{2.46}$$

with T_1, T_2, U_1 and U_2 given by (2.41)-(2.44). Now, since T_1 is geometrically distributed with parameter $1 - Q_{K_1-1}$, it immediately follows that

$$E\{T_1\} = \frac{1}{1 - Q_{K_1-1}}. \tag{2.47}$$

Moreover, using (2.14) we have that

$$\begin{aligned} E\{U_1\} &= \sum_{n=D+1}^{\infty} \Pr\{T_1 = n\} \sum_{i=1}^{n-D} E\{X_i \mid X_i < K_1\} \\ &= \sum_{n=D+1}^{\infty} Q_{K_1-1}^{n-1}(1 - Q_{K_1-1})(n-D)\frac{\sum_{k=0}^{K_1-1} kq_k}{Q_{K_1-1}} \\ &= \frac{Q_{K_1-1}^{D-1}}{1 - Q_{K_1-1}} \sum_{k=0}^{K_1-1} kq_k. \end{aligned} \tag{2.48}$$

To find expressions for $E\{T_2\}$ and $E\{U_2\}$, we need the joint distribution of

$$\Big(R_0^{(T_1)}, \dots, R_{D-2}^{(T_1)}\Big) = (X_{T_1-1}, \dots, X_{T_1-D+1}). \tag{2.49}$$

Note that $E\{T_2\}$ and $E\{U_2\}$ do not depend on $R_{D-1}^{(T_1)} = X_{T_1}$ since the critical group of customers arriving in period T_1 is always included in the batch. Next observe that $T_1 = n < D$ implies that $R_0^{(T_1)} = \cdots = R_{D-n-1}^{(T_1)} = 0$. Therefore we have to distinguish between the case where $r_0 > 0$, and the case where $r_i = 0$ for $i = 0, \ldots, m-1$ and $r_m > 0$ for some $m > 0$. For $r_0 > 0$ we have that

$$\begin{aligned}
&\Pr\{R_0^{(T_1)} = r_0, \ldots, R_{D-2}^{(T_1)} = r_{D-2}\} \\
= \;& \Pr\{X_{T_1-D+1} = r_0, \ldots, X_{T_1-1} = r_{D-2}\} \\
= \;& \sum_{n=D}^{\infty} \Pr\{T_1 = n\} \Pr\{X_{n-D+1} = r_0, \ldots, X_{n-1} = r_{D-2} \mid X_1 < K_1, \ldots, X_{n-1} < K_1, X_n \geq K_1\} \\
= \;& \sum_{n=D}^{\infty} \Pr\{T_1 = n\} \Pr\{X_{n-D+1} = r_0 \mid X_{n-D+1} < K\} \cdots \Pr\{X_{n-1} = r_{D-2} \mid X_{n-1} < K\} \\
= \;& \sum_{n=D}^{\infty} Q_{K_1-1}^{n-1}(1 - Q_{K_1-1}) \frac{q_{r_0} \cdots q_{r_{D-2}}}{Q_{K_1-1}^{D-1}} \\
= \;& \sum_{n=D}^{\infty} Q_{K_1-1}^{n-D}(1 - Q_{K_1-1}) \prod_{i=0}^{D-2} q_{r_i} \\
= \;& \prod_{i=0}^{D-2} q_{r_i} \quad (1 \leq r_0 \leq K-1;\; 0 \leq r_i \leq K-1,\; i = 1, \ldots, D-1).
\end{aligned} \tag{2.50}$$

More generally, we have for $m = 1, \ldots, D-2$ that

$$\begin{aligned}
&\Pr\{R_0^{(T_1)} = 0, \ldots, R_{m-1}^{(T_1)} = 0, R_m^{(T_1)} = r_m, \ldots, R_{D-2}^{(T_1)} = r_{D-2}\} \\
= \;& \sum_{n=D-m}^{D-1} Q_{K_1-1}^{n-1}(1 - Q_{K_1-1}) \frac{q_0^{n-D+m} q_{r_m} \cdots q_{r_{D-2}}}{Q_{K_1-1}^{n-1}} + \sum_{n=D}^{\infty} Q_{K_1-1}^{n-1}(1 - Q_{K_1-1}) \frac{q_0^m q_{r_m} \cdots q_{r_{D-2}}}{Q_{K_1-1}^{D-1}} \\
= \;& (1 - Q_{K_1-1}) \left(\prod_{i=m}^{D-2} q_{r_i}\right) \sum_{n=D-m}^{D-1} q_0^{n-D+m} + q_0^m \prod_{i=m}^{D-2} q_{r_i} \\
= \;& \prod_{i=m}^{D-2} q_{r_i} \left(q_0^m + (1 - Q_{K_1-1}) \frac{1 - q_0^m}{1 - q_0}\right) \\
& (1 \leq r_m \leq K-1;\; 0 \leq r_i \leq K-1,\; i = m+1, \ldots, D-2).
\end{aligned} \tag{2.51}$$

Note that for $m = 0$ (2.51) reduces to (2.50). Given that $(R_0^{(T_1)}, \ldots, R_{D-2}^{(T_1)}) = (r_0, \ldots, r_{D-2})$, T_2 and U_2 are just deterministic functions:

$$T_2(r_0, \ldots, r_{D-2}) := \min\Big\{D–1, \min\Big\{n = 0, \ldots, D–2 : \frac{\sum_{i=n}^{D-2} r_i}{D–n–1} \geq K_2 \text{ and } r_n \geq K_3\Big\}\Big\}; \tag{2.52}$$

$$U_2(r_0, \ldots, r_{D-2}) := \sum_{i=0}^{T_2(r_0, \ldots, r_{D-2})-1} r_i. \tag{2.53}$$

Conditioning on $(R_0^{(T_1)}, \ldots, R_{D-2}^{(T_1)})$ then yields

$$E\{T_2\} = \sum_{r_0, \ldots, r_{D-2} < K_1} \Pr\{R_0^{(T_1)} = r_0, \ldots, R_{D-2}^{(T_1)} = r_{D-2}\} T_2(r_0, \ldots, r_{D-2}); \tag{2.54}$$

$$E\{U_2\} = \sum_{r_0, \ldots, r_{D-2} < K_1} \Pr\{R_0^{(T_1)} = r_0, \ldots, R_{D-2}^{(T_1)} = r_{D-2}\} U_2(r_0, \ldots, r_{D-2}), \tag{2.55}$$

with $T_2(r_0,\ldots,r_{D-2})$ and $U_2(r_0,\ldots,r_{D-2})$ given by (2.52) and (2.53), respectively.

2.5.2 The Extended Total-Demand policy

The TD-policy only bases the batch-service decision on the total number of waiting customers, and thus completely lacks the idea of the CG-policy, i.e., the batch service should not be done when the group of customers whose delay-limit is about to expire is small. For example, when $r_0 = 0$ it is clearly better to defer the batch service for at least one period, since no immediate individual services are required. Therefore it is appropriate to extend the TD-policy in such a way that the batch-service decision does not only depend on $\sum_{i=0}^{D-1} r_i$ but also on r_0. This leads us to the Extended Total-Demand (ETD) policy: start a batch service as soon as $\sum_{i=0}^{D-1} r_i \geq K_1$ and $r_0 \geq K_2$. The cycle length of the ETD-policy is given by

$$S_{\text{ETD}} = \min\{n = 1, 2, \ldots : L_n \geq K_1 \text{ and } R_0^{(n)} \geq K_2\}. \tag{2.56}$$

The ETD-policy generalizes both the CG-policy (set $K_1 = 0$) and the TD-policy (set $K_2 = 0$), and hence must be at least as good as either of these policies. In looking for the best ETD-policy we may restrict ourselves to $K_1 \geq K_2 \geq 1$, as we know that it is suboptimal to start a batch when $r_0 = 0$. It turns out that the probabilistic computational algorithm for the TD-policy (see section 2.4) cannot be modified for the ETD-policy, except if $K_2 = 1$. To explain this, suppose that $L_n \geq K_1$. If $K_2 = 1$ then the next batch service will be started at most D periods later, namely at the first decision epoch $i \geq n$ with $R_0^{(i)} \geq K_2 = 1$ (independent of the number of arriving customers). However, if $K_2 > 1$ then this is not necessarily the case; depending on the number of arrivals, L_n may fluctuate around K_1.

Consequently, a numerical analysis can only be done by a brute-force approach similar to the one mentioned in section 2.4, in which for both $E\{S_{\text{TD}}\}$ and $E\{Y_{\text{TD}}\}$ a $(D-1)$-dimensional system of equations must be solved (see Appendix 2.B). Unfortunately, this method becomes computationally infeasible for $D \geq 5$ (say).

2.6 The optimal policy

Although the optimal policy is of little practical use, it is interesting to be able to compare the performance of the various restricted policies that we discussed in sections 2.3–2.5 with the optimal policy. As argued in section 2.1, we can compute the optimal policy by means of a Markov decision process (MDP) with state space

$$\Omega := \{\mathbf{r} \mid r_i \in \mathbb{N},\ i = 0, \ldots, D-1\}, \tag{2.57}$$

where r_i denotes the number of customers with a residual delay-limit of i periods. The action space is $\{0, 1\}$, where 1 (0) refers to the action "do (do not) start a batch service".

Define g and $v(x)$ as the expected average costs per unit of time and the relative values of an average-cost optimal policy. When taking action 0 in state $\mathbf{r}$, r_0 customers must be served individually at cost $b_I = 1$, and the next state is $(r_1, \ldots, r_{D-1}, k)$ if k customers arrive during the next period. When taking action 1 all waiting customers are served by a batch service at cost a_B, and the next state is $(0,\dots,0,k)$ if k customers arrive during the next period. Consequently, the optimality equations are given by

$$v(\mathbf{r}) = \min\Big\{r_0 - g + \sum_{k=0}^{\infty} q_k v(r_1,\dots,r_{D-1},k), a_B - g + \sum_{k=0}^{\infty} q_k v(0,\dots,0,k)\Big\} \quad (\mathbf{r}\in\Omega) \quad (2.58)$$

(see (1.12)). From [Ross 1983] (Theorem V.2.1) it follows that for this denumerable-state MDP an average-cost optimal stationary policy π^* exists. Because the dimension of Ω increases with D, the optimality equations (2.58) are only useful to compute π^* numerically for $D \le 3$. However, they can be utilized for general D to derive structural properties of the optimal policy through properties of the relative values $v(x)$.

2.6.1 Structural properties of the optimal policy

It is convenient to define

$$\begin{aligned}
h_0(\mathbf{r}) &:= r_0 - g + \sum_{k=0}^{\infty} q_k v(r_1,\dots,r_{D-1},k) \quad (\mathbf{r}\in\Omega);\\
h_1 &:= a_B - g + \sum_{k=0}^{\infty} q_k v(0,\dots,0,k);\\
h(\mathbf{r}) &:= h_1 - h_0(\mathbf{r}) \quad (\mathbf{r}\in\Omega),
\end{aligned}$$

so that the optimality equations (2.58) can be rewritten as

$$v(\mathbf{r}) = \min\{h_0(\mathbf{r}), h_1\} \quad (\mathbf{r}\in\Omega). \qquad (2.59)$$

Let $\pi^*(\mathbf{r})$ denote an optimal action in state $\mathbf{r}$, then by (2.59)

$$\pi^*(\mathbf{r}) = \begin{cases} 1 & \text{if } h(\mathbf{r}) \le 0; \\ 0 & \text{if } h(\mathbf{r}) > 0. \end{cases} \qquad (2.60)$$

The following properties of $v(\mathbf{r})$ are used to prove structural properties of an optimal policy π^*.

Theorem 2.3 *Let $\mathbf{r},\mathbf{r}' \in \Omega$ be two state vectors and let $\mathbf{e}_i$ denote the i^{th} unit vector.*
(i) If $\mathbf{r} \le \mathbf{r}'$ then $v(\mathbf{r}) \le v(\mathbf{r}')$;
(ii) $v(\mathbf{r}) - v(\mathbf{r}') \le a_B$;
(iii) $v(\mathbf{r}+\mathbf{e}_i) \le 1 + v(\mathbf{r})$ for $1 \le i \le D$;
(iv) $v(\mathbf{r}+\mathbf{e}_j) \le v(\mathbf{r}+\mathbf{e}_i$ for $1 \le i \le j \le D$;
(v) $v(\mathbf{r}) \le v(\mathbf{r}+k(\mathbf{e}_i-\mathbf{e}_j))$ for $1 \le i \le j \le D$ and $k \ge 1$;
(vi) If $r_{0,i} \le r'_{0,i}$ for all $i = 0,\dots,D-2$ and $r_{0,D-1} = r'_{0,D-1}$ then $v(\mathbf{r}) \le v(\mathbf{r}')$.

Proof. See Appendix 2.C.

The statements of Theorem 2.3 can at best be understood by thinking in terms of difference in expected future costs when starting from different states. According to (ii) this difference is always bounded by a_B, and according to (i) adding customers increases the costs, but then, according to (iii), these costs cannot increase by more than $b_I = 1$ per customer. According to (iv) the costs of adding a customer decrease with his residual delay-limit, while according to (v) moving any number of customers to a lower residual delay-limit delay increases the costs. Finally, (vi) states that if we have two states with the same total number of customers where for the first state the number of customers with a residual delay-limit of at most i is smaller for all i, then the costs in the first state are lower.

We are now ready to derive some structural properties of π^*.

Theorem 2.4 *Let $\mathbf{r} \in \Omega$ be a state vector, and let π^* be an average-cost optimal policy.*
(i) $\pi^*(0, r_1, \ldots, r_{D-1}) = 0$;
(ii) If $\pi^*(\mathbf{r}) = 1$ then $\pi^*(\mathbf{r} + \mathbf{e}_i) = 1$ for $1 \leq i \leq D$;
(iii) If $\pi^*(\mathbf{r} + \mathbf{e}_j) = 1$ then $\pi^*(\mathbf{r} + \mathbf{e}_i) = 1$ for $1 \leq i < j \leq D$.

Proof. (i) It follows from (2.58) that $\pi^*(0, r_1, \ldots, r_{D-1}) = 0$ if

$$\sum_{k=0}^{\infty} q_k \Big(v(r_1, \ldots, r_{D-1}, k) - v(0, \ldots, 0, k) \Big) \leq a_B,$$

which is true by Theorem 2.3(ii).
(ii) Using Theorem 2.3(i) we have that

$$\begin{aligned} & h(\mathbf{r} + \mathbf{e}_i) - h(\mathbf{r}) \\ = \; & h_0(\mathbf{r}) - h_0(\mathbf{r} + \mathbf{e}_i) \\ = \; & -I_{\{i=1\}} + \sum_{k=0}^{\infty} q_k \cdot \Big(v(r_1, \ldots, r_{D-1}, k) - v((r_1, \ldots, r_{D-1}, k) + I_{\{i>1\}} \mathbf{e}_{i-1}) \Big) \\ \leq \; & 0, \end{aligned}$$

where $I_{\{c\}} = 1$ if condition c is true and 0 otherwise. Since $\pi^*(\mathbf{r}) = 1$ implies $h(\mathbf{r}) \leq 0$, it follows that $h(\mathbf{r} + \mathbf{e}_i) \leq h(\mathbf{r}) \leq 0$ and hence $\pi^*(\mathbf{r} + \mathbf{e}_i) = 1$.
(iii) For $i > 1$ we use Theorem 2.3(i) and for $i = 1$ Theorem 2.3(iii) to obtain

$$\begin{aligned} & h(\mathbf{r} + \mathbf{e}_i) - h(\mathbf{r} + \mathbf{e}_j) \\ = \; & h_0(\mathbf{r} + \mathbf{e}_j) - h_0(\mathbf{r} + \mathbf{e}_i) \\ = \; & -I_{\{i=1\}} r_0' + \sum_{k=0}^{\infty} q_k \Big(v((r_1, \ldots, r_{D-1}, k) + \mathbf{e}_{j-1}) - v((r_1, \ldots, \mathbf{r}_{D-1}, k) + I_{\{i>1\}} \mathbf{e}_{i-1}) \Big) \\ \leq \; & 0. \end{aligned}$$

Since $\pi^*(\mathbf{r} + \mathbf{e}_j) = 1$ implies $h(\mathbf{r} + \mathbf{e}_j) \leq 0$, it follows that $h(\mathbf{r} + \mathbf{e}_i) \leq h(\mathbf{r} + \mathbf{e}_j) \leq 0$ and hence $\pi^*(\mathbf{r} + \mathbf{e}_i) = 1$. □

Theorem 2.4(i) confirms the fact that a batch service should not be started when the number of customers requiring an individual service is zero, because by waiting for one more period the customers arriving in the next period can be included in the batch "for free". Parts (ii) and (iii) state that if in some state the optimal decision is to start a batch, then a batch should also be started in the same state with one customer added or one customer moved to a lower residual delay-limit, respectively.

Theorem 2.4(i) has the following implication.

Corollary 2.1 $S_{\pi^*} \geq D$.

We close this subsection with two "second-order" results.

Theorem 2.5 *Let* $\mathbf{r} \in \Omega$ *be a state vector.*
(i) $v(\mathbf{r}+\mathbf{e}_i+\mathbf{e}_j) - v(\mathbf{r}+\mathbf{e}_i) \leq v(\mathbf{r}+\mathbf{e}_j) - v(\mathbf{r})$ *for* $1 \leq i,j \leq D$*;*
(ii) $v(\mathbf{r}-\mathbf{e}_j+\mathbf{e}_{j-k}) \leq v(\mathbf{r}-\mathbf{e}_i+\mathbf{e}_{i-k})$ *for* $k < i \leq j \leq D$*.*

Proof. See Appendix 2.C.

Theorem 2.5(i) states that the cost of adding a customer decreases with the number of additional customers, while (ii) states that the cost of decreasing the residual delay-limit of a customer by k periods is decreasing in his initial residual delay-limit.

2.6.2 Solution for the case $D = 2$

We now use the optimality equations (2.58) and some of the properties of the optimal policy (see Theorem 2.4) for a detailed solution procedure for the special case $D = 2$. In this case the state space is

$$\Omega := \{(i,j) \mid i,j \in I\!N\}, \tag{2.61}$$

with i (j) denoting the number of customers with a residual delay-limit of 0 (1). It follows from Theorem 2.4(ii) that the optimal policy has the following general structure:

$$\pi^*(i,j) = \begin{cases} 0 & \text{if } i < K_j^*; \\ 1 & \text{if } i \geq K_j^*. \end{cases} \tag{2.62}$$

By appropriately choosing K_j ($j = 0,1,\ldots$) we see that (2.62) includes the following restricted policies.
(a) $K_j = K$ for all j: CG-policy (section 2.3);
(b) $K_j = K - j$ for all j: TD-policy (section 2.4), but without the stipulation that $S_\pi \geq D$;
(c) $K_j = \max\{K_1 - j, K_2\}$: ETD-policy (section 2.5.2).

Theorem 2.6 *Let* $(K_0^*, K_1^*, \ldots)$ *be an average-cost optimal policy for* $D = 2$*.*
(i) $K_0^* = \lceil a_B \rceil$*;*
(ii) K_j^* *is non-increasing in* j *for* $j = 0,1,\ldots$*;*
(iii) $K_j^* - K_{j+1}^* \in \{0,1\}$ *for* $j = 0,1,\ldots$*.*

Proof. (i) It follows from (2.58) that $\pi^*(i,0) = 1$ if $i > 0$ and $a_B < i$, implying that $K_0^* = \min\{i : a_B < i\} = \lceil a_B \rceil$.
(ii) Suppose that $K_j^* < K_{j+1}^*$ for some j. Then $\pi^*(K_j^*, j) = 1$ and $\pi^*(K_j^*, j+1) = 0$, contradicting Theorem 2.4(ii).
(iii) It follows from (ii) that $K_j^* - K_{j+1}^* \geq 0$. Therefore suppose that $K_j^* - K_{j+1}^* > 1$

for some j. Then $\pi^*(K^*_{j+1}, j+1) = 1$ while $\pi^*(K^*_{j+1}+1, j) = 0$ since $K^*_{j+1}+1 < K^*_j$, contradicting Theorem 2.4(iii). □

For a fixed policy $\pi = (K_1, K_2, \ldots)$ the expected average costs g_π and the relative values $v_\pi(i,j)$ can be calculated by solving the following system:

$$\begin{aligned} v_\pi(i,j) &= i - g_\pi + \sum_{k=0}^{\infty} q_k v_\pi(j,k) \quad (i < K_j); \\ v_\pi(i,j) &= a_B - g_\pi + \sum_{k=0}^{\infty} q_k v_\pi(0,k) \quad (i \geq K_j); \\ v_\pi(0,0) &= -g_\pi + \sum_{k=0}^{\infty} q_k v_\pi(0,k) := 0 \end{aligned} \tag{2.63}$$

(see (1.13)). It immediately follows from (2.63) that

$$v_\pi(i,j) = \begin{cases} i & \text{if } j = 0 \text{ and } 0 \leq i < K_0; \\ i + v_\pi(0,j) & \text{if } j > 0 \text{ and } 0 \leq i < K_j; \\ a_B & \text{if } j \geq 0 \text{ and } i \geq K_j. \end{cases} \tag{2.64}$$

It remains to find $v_\pi(0,j)$ for $j > 0$, and by using (2.63) and (2.64) we obtain

$$\begin{aligned} v_\pi(0,j) &= -g_\pi + \sum_{k: j < K_k} q_k\big(j + v_\pi(0,k)\big) + \sum_{k: j \geq K_k} q_k a_B \\ &= a_B - g_\pi + \sum_{k=0}^{K_j^{-1}-1} q_k\big(j - a_B + v_\pi(0,k)\big) \\ &= a_B - g_\pi + (j - a_B) Q_{K_j^{-1}-1} + \sum_{k=0}^{K_j^{-1}-1} q_k v_\pi(0,k) \quad (j \geq 0). \end{aligned} \tag{2.65}$$

Here

$$K_j^{-1} := \begin{cases} \infty & \text{if } 0 \leq j < K_\infty; \\ \min\{i : K_i \leq j\} & \text{if } j \geq K_\infty, \end{cases} \tag{2.66}$$

so that $K_j^{-1} = i$ if and only if $K_i \leq j$ and $K_{i-1} > j$. For example, if $K_0 = 4, K_1 = 3, K_2 = 3, K_3 = \cdots = K_\infty = 2$, then $K_0^{-1} = K_1^{-1} = \infty, K_2^{-1} = 3, K_3^{-1} = 1, K_4^{-1} = \cdots = K_\infty^{-1} = 0$. Note that K_j^{-1} is well-defined by Theorem 2.6(ii) and strictly decreasing in j for $j \geq K_\infty$ by Theorem 2.6(iii). Now we have reduced the two-dimensional system (2.63) for $v_\pi(i,j)$ and g_π to the one-dimensional system (2.65) for $v_\pi(0,j)$ and g_π, although the number of equations in (2.65) is infinite. However, since $j < K_\infty$ iff $K_j^{-1} = \infty$ and $j \geq K_0$ iff $K_j^{-1} = 0$ we have that

$$v_\pi(0,j) = \begin{cases} j & \text{if } 0 \leq j < K_\infty; \\ a_B - g_\pi & \text{if } j \geq K_0. \end{cases} \tag{2.67}$$

Using (2.67) we can solve (2.65) for those j with $K_j^{-1} \leq K_\infty$, i.e., for $j \geq K_{K_\infty}^{-1}$, yielding

$$v_\pi(0,j) = a_B - g_\pi + (j - a_B) Q_{K_j^{-1}-1} + \sum_{k=0}^{K_j^{-1}-1} k q_k \quad (K_{K_\infty}^{-1} \leq j < K_0). \tag{2.68}$$

Consequently, for policies with $K^{-1}_{K_\infty} \leq K_\infty$, (2.67) and (2.68) together give a complete solution of (2.65). Moreover, substituting this solution into

$$g_\pi = \sum_{k=0}^{\infty} q_k v_\pi(0,k) \tag{2.69}$$

and solving for g_π, we obtain

$$g_\pi = \frac{a_B(1-Q_{K_\infty-1}) + \sum_{j=1}^{K_\infty-1} jq_j + \sum_{j=K_\infty}^{K_0-1}(j-a_B)q_jQ_{K_j^{-1}-1} + \sum_{j=K_\infty}^{K_0-1} q_j \sum_{k=0}^{K_j^{-1}-1} kq_k}{2-Q_{K_\infty-1}}. \tag{2.70}$$

On the other hand, if $K^{-1}_{K_\infty} > K_\infty$ then what remains of (2.65) are the equations for $K_\infty \leq j < K^{-1}_{K_\infty}$, and using (2.67) we find that

$$v_\pi(0,j) = a_B - g_\pi + (j-a_B)Q_{K_j^{-1}-1} + \sum_{k=0}^{K_\infty-1} kq_k + \sum_{k=K_\infty}^{K_j^{-1}-1} q_k v_\pi(0,k) \quad (K_\infty \leq j < K^{-1}_{K_\infty}). \tag{2.71}$$

Finally, substituting (2.67) and (2.68) into (2.69), we end up with a finite system of $K^{-1}_{K_\infty} - K_\infty + 1$ equations in the unknowns $v_\pi(0,j)$ $(j = K_\infty, \ldots, K^{-1}_{K_\infty} - 1)$ and g_π. The reduced system (2.71) can be solved very efficiently; this is useful when computing the optimal policy by means of a policy iteration algorithm (see section 1.5), where in every step system (2.63) must be solved for some policy π.

2.7 Numerical comparisons

Table 2.1 provides the expected average costs and the optimal control-limit(s) for the Never-Batch (NB), Only-Batch (OB), Critical-Group (CG), Extended Critical-Group (ECG), Total-Demand (TD) and Extended Total-Demand (ETD) policies, as well as the expected average costs of the optimal policy, for the case $D = 2$. We use a Poisson distribution $q_k = e^{-\lambda}\frac{\lambda^k}{k!}$ for $\lambda \in \{1,3,5,10\}$ and $a_B = c\lambda D$ $(c \in \{0.75, 1, 1.25\})$, making a total of 12 instances. The corresponding optimal policies, characterized by the control-limits $(K_i^*,\ i = 0,1,\ldots)$ (see (2.62)), are given in Table 2.2. We use the following shorthand notation: n^m denotes a string of m n's, n-m denotes the string $n, n-1, \ldots, m$ $(m < n)$ and the last number is K^*_∞.

It turns out that for $D = 2$ the ETD-policy performs extremely well; in 5 out of 12 cases the optimal policy coincides with the ETD-policy, while in the other cases the difference in costs is less than 1 percent. Also, the CG-policy outperforms the TD-policy in most cases, while the ECG-policy considerably improves upon the CG-policy (except for $\lambda = 3$ and $a_B = 4.5$, where they are the same).

Remark. The global optimal policy is not uniquely determined; for example, if a_B is integer-valued and $D = 2$ then the policies with $K_0 = a_B$ and $K_0 = a_B + 1$ have exactly the same cost (see (2.58) and Theorem 2.6(i)). This explains the cases in Table 2.1 where

λ	a_B	g_{NB}	g_{OB}	$g_{\mathrm{CG}}(K^*)$	$g_{\mathrm{ECG}}(K_1^*, K_2^*)$	$g_{\mathrm{TD}}(K^*)$	$g_{\mathrm{ETD}}(K_1^*, K_2^*)$	g_{π^*}
1	1.5	1	0.5810	0.5810 (1)	0.5716 (2,1)	0.6138 (2)	0.5395 (2,1)	0.5395
	2	1	0.7746	0.7090 (2)	0.6848 (2,1)	0.7335 (3)	0.6848 (3,1)	0.6848
	2.5	1	0.9683	0.8135 (2)	0.7980 (2,1)	0.8311 (3)	0.7797 (3,1)	0.7797
3	4.5	3	2.1926	2.0250 (3)	2.0250 (3,3)	2.1398 (5)	2.0012 (5,3)	2.0012
	6	3	2.9234	2.5031 (4)	2.4723 (4,3)	2.5862 (7)	2.4438 (7,3)	2.4438
	7.5	3	3.6543	2.8084 (5)	2.7680 (5,3)	2.8169 (9)	2.7303 (8,4)	2.7275
5	7.5	5	3.7373	3.5364 (4)	3.5096 (5,4)	3.6650 (8)	3.4921 (8,4)	3.4921
	10	5	4.9831	4.3661 (6)	4.3337 (6,5)	4.4838 (12)	4.2803 (11,5)	4.2803
	12.5	5	6.2289	4.8334 (8)	4.7806 (8,5)	4.8323 (14)	4.7299 (13,6)	4.7288
10	15	10	7.4998	7.3032 (8)	7.2918 (9,8)	7.4509 (15)	7.2762 (15,8)	7.2762
	20	10	9.9998	9.1171 (11)	9.0479 (12,10)	9.2786 (22)	8.9814 (21,10)	8.9814
	25	10	12.4997	9.9013 (16)	9.8427 (15,10)	9.8716 (27)	9.7744 (26,11)	9.7743

Table 2.1: Numerical comparison of different policies for $D = 2$

λ	a_B	g_{π^*}	$(K_0^*, K_1^*, \ldots)$
1	1.5	0.5395	(2,1)
	2	0.6848	$(2^2,1)$
	2.5	0.7797	(3-1)
3	4.5	2.0012	(5-3)
	6	2.4438	$(6,5,4^2,3)$
	7.5	2.7275	$(8\text{-}5,4^2,3)$
5	7.5	3.4921	(8-4)
	10	4.2803	$(10\text{-}7,6^2,5)$
	12.5	4.7288	$(13\text{-}7,6^3,5)$
10	15	7.2762	(15-9,8)
	20	8.9814	$(20\text{-}13,12^2,11,10^4,9)$
	25	9.7743	$(25\text{-}17,16^2,15\text{-}13,12^2,11^3,10)$

Table 2.2: The optimal policy for $D = 2$

λ	a_B	g_{NB}	g_{OB}	$g_{\mathrm{CG}}(K^*)$	$g_{\mathrm{ECG}}(K_1^*, K_2^*, K_3^*)$	$g_{\mathrm{TD}}(K^*)$	$g_{\mathrm{ETD}}(K_1^*, K_2^*)$	g_{π^*}
1	2.25	1	0.6281	0.6281 (1)	0.5944 (2,1,1)	0.6310 (3)	0.5843 (3,1)	0.5798
	3	1	0.8375	0.7593 (2)	0.7364 (2,1,1)	0.7551 (4)	0.7270 (4,1)	0.7229
	3.75	1	1.0469	0.8890 (2)	0.8643 (3,1,1)	0.8467 (5)	0.8339 (5,1)	0.8253
3	6.75	3	2.2114	2.0853 (3)	2.0853 (3,3,3)	2.1275 (8)	2.0589 (7,3)	2.0537
	9	3	2.9485	2.6059 (4)	2.5638 (5,3,3)	2.5734 (11)	2.5215 (10,3)	2.5157
	11.25	3	3.6856	2.9027 (6)	2.8520 (6,3,3)	2.8240 (13)	2.8021 (12,4)	2.7988
5	11.25	5	3.7415	3.5958 (5)	3.5725 (5,4,4)	3.6459 (13)	3.5625 (12,4)	3.5523
	15	5	4.9887	4.5038 (6)	4.4283 (7,4.5,5)	4.4428 (17)	4.3815 (16,5)	4.3739
	18.75	5	6.2359	4.9375 (9)	4.8786 (9,5,5)	4.8323 (20)	4.8156 (20,6)	4.8090
10	22.5	10	7.4999	7.3632 (8)	7.3499 (9,7.5,8)	7.4419 (25)	7.3437 (23,8)	
	30	10	9.9998	9.2920 (12)	9.2061 (13,9.5,10)	9.2114 (33)	9.1251 (31,10)	
	37.5	10	12.4998	9.9800 (18)	9.9412 (17,10,10)	9.8757 (39)	9.8672 (38,12)	

Table 2.3: Numerical comparison of different policies for $D = 3$

the cost of the optimal ETD-policy is equal to the cost of the global optimal policy, while the optimal policy given in Table 2.2 is not of the ETD-type. In these cases the optimal ETD-policy may well be an alternative global optimal policy.

Table 2.3 repeats the above calculations for the case $D = 3$, except that we do not give the form of the optimal policy because of its complex structure (the minimum cost for $\lambda = 10$ is omitted due to computational infeasibility). Although in none of the cases the optimal policy coincides with the ETD-policy, the ETD-policy is always close to optimal and performs best of the restricted policies. Except for low values of λ and a_B the TD-policy now performs considerably better than the CG-policy.

It is also interesting to graphically compare the performance of the various policies as a function of the standardized cost parameter a_B. In Figure 2.1 we plot the average costs of the optimal policy and the best policy within the restricted classes as a function of a_B for the case $D = 2$, $\lambda = 3$. The two extreme policies, the NB- (Never Batch) and OB- (Only Batch) policy with respective cost functions given by (2.4) and (2.8), correspond to the straight lines in the graph. Next, in Figure 2.2, we plot the percentage savings of the optimal policy and the various restricted policies with respect to the best of the NB- and the OB-policy as a function of a_B, thereby establishing the value of using these more sophisticated policies. Finally, in Figures 2.3 and 2.4 we do the same for the case $D = 3$, $\lambda = 2$.

In view of Figures 2.1 and 2.3, the cost functions of the optimal policy as well as the restricted policies appear to be concave in a_B. For small values of a_B the OB-policy performs well since $\lim_{a_B \to 0} g_{\pi^*} = g_{\text{OB}}$, while for large values of a_B the NB-policy performs well since $\lim_{a_B \to \infty} g_{\pi^*} = g_{\text{NB}}$. Therefore, using the best of the NB- and OB-policy (denoted by NB$\wedge$OB) is a reasonable simple policy that does not use detailed information on the state of the system. It is easily seen that for $a_B = \hat{a}_B := b_I \mu (D + \frac{q_0}{1-q_0})$ the NB- and OB-policy coincide, whence for $a_B < \hat{a}_B$ ($a_B > \hat{a}_B$) the OB-policy (NB-policy) is better. Figures 2.2 and 2.4 reveal that the savings of the various policies with respect to the NB$\wedge$OB-policy, given by $\frac{\min\{g_{\text{NB}}, g_{\text{OB}}\} - g_\pi}{\min\{g_{\text{NB}}, g_{\text{OB}}\}}$, are maximal around $\hat{a}_B$ where $g_{\text{NB}} = g_{\text{OB}}$. These graphs also make it possible to order the performance of the different policies for any value of a_B. In particular, we note that for small values of a_B the TD-policy is even worse than the NB$\wedge$OB-policy (as it can trigger a batch service when no individual services are needed), while for increasing values of a_B the TD-policy at some point becomes better than both the CG- and the ECG-policy. For $D = 3$ this critical value is larger than for $D = 2$, reflecting the fact that the performance of the TD-policy relative to the CG-policy increases with D. Moreover, it is intuitively clear that the relative performance of the CG-policy decreases with D, as the significance of the critical group decreases.

2.8 The role of D

Because the delay-limit D determines the dimension of the state space (see (2.57)), most of the numerical results in this chapter are limited to moderate values of D. As a matter of fact, the only policies discussed here that do not suffer from the curse of dimensionality are the Never-Batch and Only-Batch policy (see section 2.2), as well as the Critical-Group

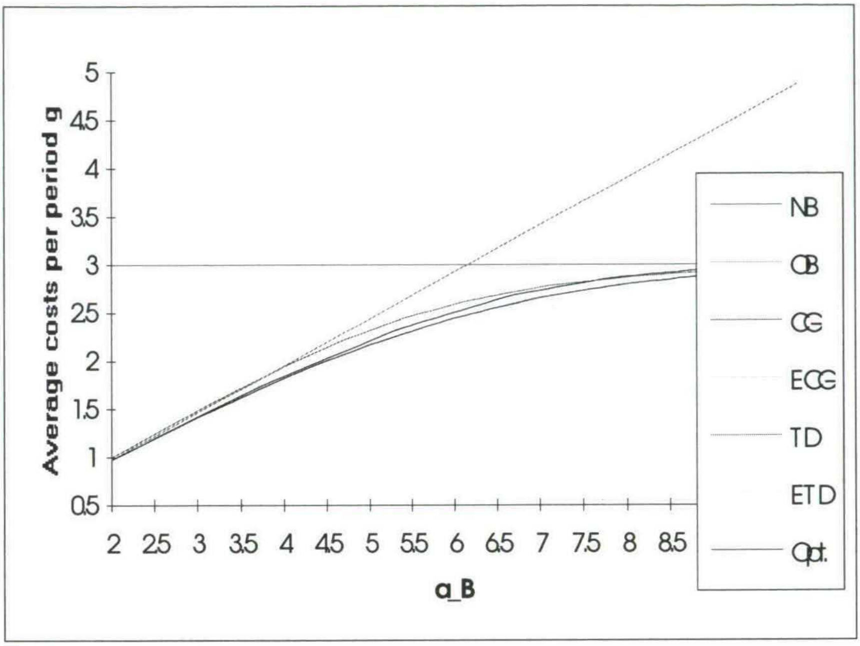

Figure 2.1: Expected average costs per period ($D = 2$, $\lambda = 3$)

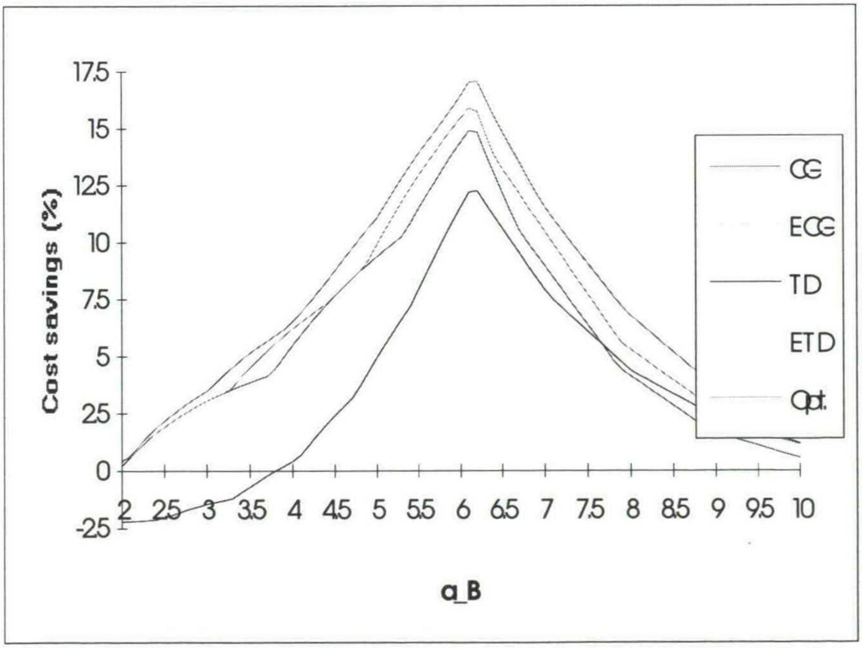

Figure 2.2: Percentage cost savings ($D = 2$, $\lambda = 3$)

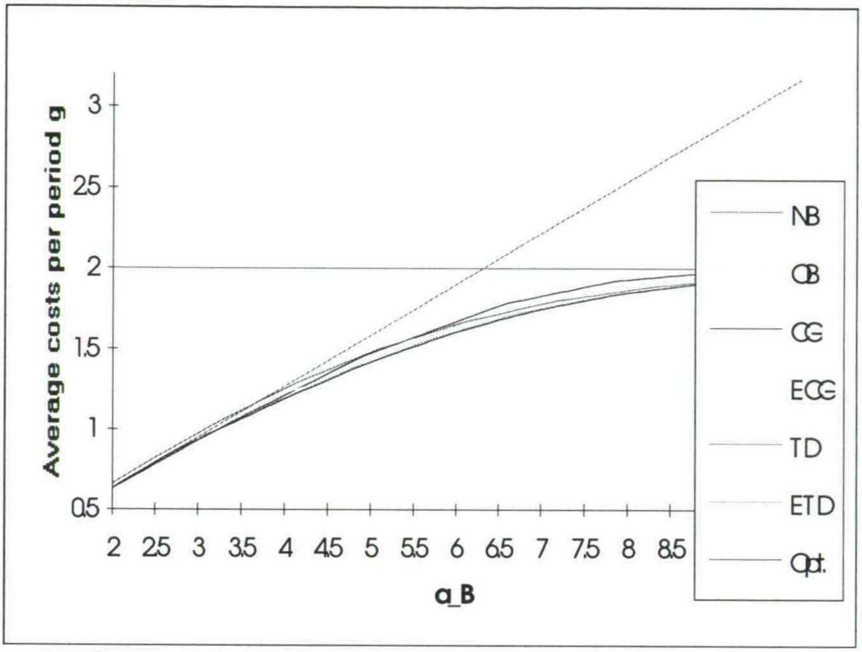

Figure 2.3: Expected average costs per period ($D = 3$, $\lambda = 2$)

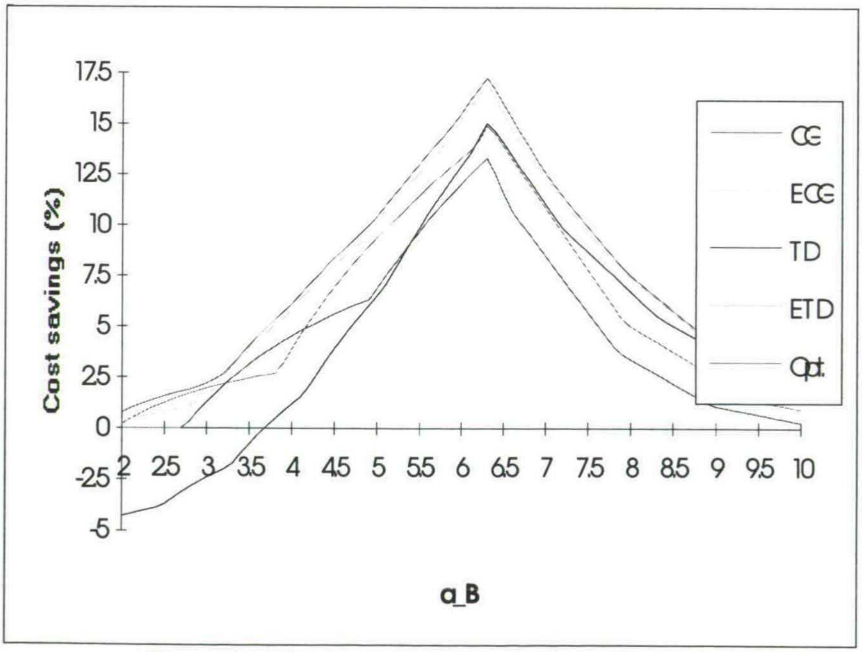

Figure 2.4: Percentage cost savings ($D = 3$, $\lambda = 2$)

policy (see section 2.3). The CG-policy is only based on the number of customers in a single period, which reduces the dimension of the state space to 1. Unfortunately though, as noted in the previous section, the performance of the CG-policy decreases rapidly with D. On the other hand, the performance of the Extended Critical-Group policy increases with D, as there is more freedom in the exact timing of batch service (after arrival of the critical group). Since the ECG-policy only becomes computationally infeasible for $D > 10$ (say), it is a good alternative for somewhat larger values of D.

We will now argue that there is no need to use larger values of D, and that the value of D can be used as a decision variable that determines the accuracy of the model. Simply observe that the value of D can be decreased by increasing the period length t accordingly, while keeping the absolute delay-limit constant. Specifically, suppose that the absolute delay-limit equals D_0 periods for $t_0 = 1$. Then we can use any policy for $D < D_0$ by setting $t = D_0/D$, or $D_0 = D \cdot t$. Here it depends on the specific application whether t can take on any positive value, or is restricted to integer multiples of t_0 (e.g., days or weeks). Obviously, changing the period length also changes the distribution of the number of customers per period (X_n). If X_n has a Poisson distribution, then changing the period length only requires a proportional change in the parameter (λt). For other initial distributions the distribution itself may also change, depending on whether or not the class of distributions is closed with respect to taking convolutions. It is also important that $\{X_n\}$ remains a sequence of i.i.d. random variables.

To compare policies for different values of D we have to adjust g_π for the period length t, i.e., the expected average costs per period of length t_0 are given by $\frac{g_\pi}{t}$. Since increasing the period length is equivalent to decreasing the frequency of decision epochs, the expected average costs of any policy (per period of length t_0) decrease with D. Moreover, it turns out that the expected average costs are concave in D, i.e., the marginal gain is a decreasing function of D. When starting with $D = 1$ and $t = D_0$, the step from $D = 1$ to $D = 2$ provides the largest gain, followed by the step from $D = 2$ to $D = 3$. Increasing D further does usually not lead to significant additional savings, so that the policies for $D = 3$ are satisfactory for any value of $D_0 > D$. As a result, it is not very restrictive that most policies are only feasible for small values of D.

To illustrate, suppose that $t_0 = 1$ (day), $D_0 = 3$ (days) and that the number of customers per day has a Poisson distribution with mean λ_0. Then we can use any policy π for $D \leq D_0$ by setting $t = \frac{D_0}{D}$ and $\lambda = \frac{\lambda_0 D_0}{D}$. Table 2.4 gives the global minimal expected average costs per day for $\lambda_0 \in \{1, 3, 5\}$ and $D \in \{1, 2, 3\}$. We see that the difference in costs between $D = 1$ and $D = 2$ is larger than between $D = 2$ and $D = 3$.

Since D can be treated as a decision variable, all heuristic policies in this chapter can also be optimized with respect to D. It is very well possible that using a value of $D < D_0$ leads to better results in terms of expected average costs (per period of length t_0). For most policies D is limited to small values, but for the CG-policy there is no restriction. Therefore the CG-policy can be employed as a two-parameter policy: the average-cost optimal CG-policy follows from

$$\min_{D=1,\dots,D_0} \Big\{ \min_{K=1,2,\dots} \frac{D}{D_0} g_{\text{CG}}(K) \Big\}, \tag{2.72}$$

where $g_{\text{CG}}(K)$ is given by (2.19). In Table 2.5 we consider the case $D_0 = 10$ (days), $t_0 = 1$

λ_0	a_B	D	t	λ	$\frac{1}{t}g_{\pi^*}$
1	3	1	3	3	0.7760
		2	1.5	1.5	0.7420
		3	1	1	0.7229
3	9	1	3	9	2.6047
		2	1.5	4.5	2.5468
		3	1	3	2.5157
5	15	1	3	15	4.4878
		2	1.5	7.5	4.4123
		3	1	5	4.3739

Table 2.4: $\frac{g_{\pi^*}}{t}$ is concave in D

D	t	λ	K^*	$\frac{1}{t}g_{CG}(K^*)$
1	10	100	100	9.6014
2	5	50	52	9.6076
3	3.3333	33.3333	36	9.6177
4	2.5	25	28	9.6273
5	2	20	23	9.6368
6	1.6667	16.6667	20	9.6434
7	1.4286	14.2857	18	9.6508
8	1.25	12.5	16	9.6565
9	1.1111	11.1111	15	9.6645
10	1	10	13	9.6704

Table 2.5: $g_{CG}(K^*)$ as a function of D

(day), a Poisson(λt) distributed number of customers per period of length t, $\lambda_0 = 10$ and $a_B = 100$, and we compute $g_{CG}(K^*)$ for $D = 1, \ldots, D_0$. Note that $g_{CG}(K)$ for $D = 1$ reduces to (2.1). For this particular example with a delay-limit of 10 weeks, it is better to "inspect" every 10 weeks and start a batch service depending on the total number of customers, than to inspect every week and start a batch service depending on the size of the critical group.

Appendix 2.A: Proof of Theorem 2.1

In this appendix we show the optimality of a control-limit type policy within the class of Critical-Group policies. Define for $n = 1, 2, \ldots$

$$\begin{aligned} T_n &:= \text{number of periods since the last batch service at the end of period } n; \\ U_n &:= \begin{cases} T_n & \text{if } T_n < D; \\ R_0^{(n)} & \text{if } T_n \geq D. \end{cases} \end{aligned}$$

Then $\{U_n,\ n = 1, 2, \ldots\}$ is a stochastic process on the state space

$$\{i' \mid i' = 1, \ldots, D-1\} \bigcup \{i \mid i = 0, 1, \ldots\}. \tag{2.73}$$

Finding the optimal policy within the class of CG-policies boils down to solving the following optimality equations:

$$\begin{aligned} v(i') &= -g + v((i+1)') \quad (i' = 1, \ldots, D-2); \\ v(D-1') &= -g + \sum_{k=0}^{\infty} q_k v(k); \\ v(i) &= \min\Big\{a_B - g + v(1'), i - g + \sum_{k=0}^{\infty} q_k v(k)\Big\} \quad (i = 0, 1, \ldots). \end{aligned} \tag{2.74}$$

It is easily seen that

$$v(1') = -(D-2)g + v((D-1)') = -(D-1)g + \sum_{k=0}^{\infty} q_k v(k), \tag{2.75}$$

so that (2.74) reduces to

$$v(i) = \min\Big\{a_B - Dg + \sum_{k=0}^{\infty} q_k v(k), i - g + \sum_{k=0}^{\infty} q_k v(k)\Big\} \quad (i = 0, 1, \ldots). \tag{2.76}$$

It follows from (2.76) that a batch service is started if $i > a_B - (D-1)g$, proving the control-limit structure.

The optimality equations (2.74) can also be used to verify (2.19). For a fixed CG-policy with control-limit K we have that

$$v_{CG}(i) = \begin{cases} i - g_{CG} + \sum_{k=0}^{\infty} q_k v_{CG}(k) & \text{if } i < K; \\ a_B - Dg_{CG} + \sum_{k=0}^{\infty} q_k v_{CG}(k) & \text{if } i \geq K, \end{cases} \tag{2.77}$$

which upon setting $v_{CG}(0) = 0$ reduces to

$$v_{CG}(i) = \begin{cases} i & \text{if } i < K; \\ a_B - (D-1)g_{CG} & \text{if } i \geq K. \end{cases} \tag{2.78}$$

Next we find g_{CG} from

$$g_{\text{CG}} = \sum_{k=0}^{\infty} q_k v_{\text{CG}}(k) = \sum_{k=0}^{K-1} k q_k + \big(a_B - (D-1) g_{\text{CG}}\big)(1 - Q_{K-1}), \tag{2.79}$$

yielding

$$g_{\text{CG}} = \frac{a_B(1 - Q_{K-1}) + \sum_{k=0}^{K-1} k q_k}{Q_{K-1} + D(1 - Q_{K-1})},$$

in accordance with (2.19).

Appendix 2.B: A brute-force method

In this appendix we present a brute-force method to compute $E\{S_\pi\}$ and $E\{Y_\pi\}$ for the TD- and the ETD-policy, by computing first entrance times and "costs" for the Markov chain $\{\mathbf{R}^{(n)}\}$ into the sets $\{\mathbf{r} : r_{0,D-1} \geq K\}$ and $\{\mathbf{r} : r_{0,D-1} \geq K_1 \text{ and } r_0 \geq K_2\}$, respectively.

We start with the TD-policy. Define

$S_{\text{TD}}(r_1, \ldots, r_{D-1})$:= expected number of periods until the next batch service starting with r_i customers with a residual delay-limit of i periods ($i = 1, \ldots, D-1$), given that there is no immediate batch service;

$Y_{\text{TD}}(r_1, \ldots, r_{D-1})$:= expected number of individual services until the next batch service starting with r_i customers with a residual delay-limit of i periods ($i = 1, \ldots, D-1$), given that there is no immediate batch service and excluding possible immediate individual services.

Note that these quantities are defined in such a way that they are independent of r_0, thereby reducing the dimension of the state space from D to $D-1$. Conditioning on the number of arriving customers in the next period we obtain the following two finite systems of equations for $r_{1,D-1} < K$:

$$S_{\text{TD}}(r_1, \ldots, r_{D-1}) = 1 + \sum_{k=0}^{K-1-r_{1,D-1}} q_k S_{\text{TD}}(r_2, \ldots, r_{D-1}, k); \tag{2.80}$$

$$Y_{\text{TD}}(r_1, \ldots, r_{D-1}) = \sum_{k=0}^{K-1-r_{1,D-1}} q_k \big(r_1 + Y_{\text{TD}}(r_2, \ldots, r_{D-1}, k)\big). \tag{2.81}$$

Finally, incorporating the stipulation that $S_{\text{TD}} \geq D$, we have that

$$E\{S_{\text{TD}}\} = D + \sum_{\substack{k_1,\ldots,k_D: \\ k_{1,D} < K}} q_{k_1} \cdots q_{k_D} S_{\text{TD}}(k_2, \ldots, k_D); \tag{2.82}$$

$$E\{Y_{\text{TD}}\} = \sum_{\substack{k_1,\ldots,k_D: \\ k_{1,D} < K}} q_{k_1} \cdots q_{k_D} \big(k_1 + Y_{\text{TD}}(k_2, \ldots, k_D)\big) = Y_{\text{TD}}(0, \ldots, 0). \tag{2.83}$$

To set up a finite system of equations for the ETD-policy is more complicated. We exploit the fact that batch services are now limited to periods with $R_0^{(n)} \geq K_2$, i.e., we use the Markov chain $\{\mathbf{R}^{(n)}\}$ embedded on $\{\mathbf{r} : r_0 \geq K_2\}$. Define

$S_{\text{ETD}}(r_1, \dots, r_{D-1}) :=$ expected number of periods until the next batch service starting with r_i customers with a residual delay-limit of i periods ($i = 1, \dots, D-1$), given that there is no immediate batch service and $r_0 \geq K_2$;

$Y_{\text{ETD}}(r_1, \dots, r_{D-1}) :=$ expected number of individual services until the next batch service starting with r_i customers with a residual delay-limit of i periods ($i = 1, \dots, D-1$), given that there is no immediate batch service and $r_0 \geq K_2$, and excluding possible immediate individual services.

For a given state $(r_1, \dots, r_{D-1})$ with $r_{1,D-1} < K_1$ let j be the smallest integer for which $r_j \geq K_2$. For states with $j \leq D-1$, conditioning on the number of customers in the next j periods yields

$$S_{\text{ETD}}(r_1, \dots, r_{D-1}) = j + \sum_{\substack{k_1,\dots,k_j: \\ k_{1,j} < K_1 - r_{j,D-1}}} q_{k_1} \cdots q_{k_j} S_{\text{ETD}}(r_{j+1}, \dots, r_{D-1}, k_1, \dots, k_j); \tag{2.84}$$

$$Y_{\text{ETD}}(r_1, \dots, r_{D-1}) = r_{1,j-1} + \sum_{\substack{k_1,\dots,k_j: \\ k_{1,j} < K_1 - r_{j,D-1}}} q_{k_1} \cdots q_{k_j} \Big(r_j + Y_{\text{ETD}}(r_{j+1}, \dots, r_{D-1}, k_1, \dots, k_j) \Big). \tag{2.85}$$

On the other hand, for states $(r_1, \dots, r_{D-1})$ with $r_{1,D-1} < K_1$ and $r_i < K_2$ for all $i = 1, \dots, D-1$, conditioning on the number of customers until the first decision epoch n with $R_0^{(n)} \geq K_2$ yields

$$S_{\text{ETD}}(r_1, \dots, r_{D-1}) = \frac{1}{1 - Q_{K_2-1}} + D - 1 + \sum_{\substack{k_1,\dots,k_{D-1}: \\ k_{1,D-1} < K_1}} q_{k_1} \cdots q_{k_{D-1}} S_{\text{ETD}}(k_1, \dots, k_{D-1}); \tag{2.86}$$

$$Y_{\text{ETD}}(r_1, \dots, r_{D-1}) = r_{1,D-1} + \frac{\sum_{k=0}^{K_2-1} k q_k}{1 - Q_{K_2-1}} + \sum_{\substack{k_0,\dots,k_{D-1}: \\ K_2 \leq k_0 < K_1 \\ k_{1,D-1} < K_1 - k_0}} q_{k_0} \cdots q_{k_{D-1}} \Big(k_0 + Y_{\text{ETD}}(k_1, \dots, k_{D-1}) \Big). \tag{2.87}$$

Finally, we have that

$$E\{S_{\text{ETD}}\} = S_{\text{ETD}}(0, \dots, 0); \tag{2.88}$$
$$E\{Y_{\text{ETD}}\} = Y_{\text{ETD}}(0, \dots, 0). \tag{2.89}$$

Appendix 2.C: Proofs of Theorems 2.3 and 2.5

In this appendix we present the proofs of Theorems 2.3 and 2.5. We prove all statements for the finite-horizon discounted-cost counterpart; the corresponding statements for the average-cost case considered here follow by letting the horizon length tend to infinity and the discount factor to one, and applying limit theorems from dynamic programming (see [Ross 1983] for details). Define for $n = 1, 2, \ldots$

$$
\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}) &:= \text{minimal } \alpha\text{-discounted costs starting in state } \mathbf{r} \text{ with } n \text{ periods left;}\\
\pi_n^*(\mathbf{r}) &:= \text{optimal action in state } \mathbf{r} \text{ with } n \text{ transitions to go;}\\
h_{n,0}^{(\alpha)}(\mathbf{r}) &:= r_0 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}(r_1, \ldots, r_{D-1}, k) \quad (\mathbf{r} \in \Omega);\\
h_{n,1}^{(\alpha)} &:= a_B + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}(0, \ldots, 0, k);\\
h_n^{(\alpha)}(\mathbf{r}) &:= h_{n,1}^{(\alpha)} - h_{n,0}^{(\alpha)}(\mathbf{r}) \quad (\mathbf{r} \in \Omega).
\end{aligned}
$$

Proof of Theorem 2.3.
(i) Obviously,

$$v_1^{(\alpha)}(\mathbf{r}) = \min\{r_0, a_B\} \le \min\{r_0', a_B\} = v_1^{(\alpha)}(\mathbf{r}').$$

Next, using the induction hypothesis,

$$
\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}) &= \min\Big\{r_0 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}(r_1, \ldots, r_{D-1}, k), h_{n,1}^{(\alpha)}\Big\}\\
&\le \min\Big\{r_0' + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}(r_1', \ldots, r_{D-1}', k), h_{n,1}^{(\alpha)}\Big\}\\
&= v_n^{(\alpha)}(\mathbf{r}').
\end{aligned}
$$

(ii) Using (i) we have that

$$
\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}) - v_n^{(\alpha)}(\mathbf{r}') &= \min\{h_{n,0}^{(\alpha)}(\mathbf{r}), h_{n,1}^{(\alpha)}\} - \min\{h_{n,0}^{(\alpha)}(\mathbf{r}'), h_{n,1}^{(\alpha)}\}\\
&\le \max\{0, h_{n,1}^{(\alpha)} - h_{n,0}^{(\alpha)}(\mathbf{r}')\}\\
&= \max\Big\{0, a_B - r_0' - \alpha \sum_{k=0}^{\infty} q_k \Big(v_{n-1}^{(\alpha)}(r_1', \ldots, r_{D-1}', k) - v_{n-1}^{(\alpha)}(0, \ldots, 0, k)\Big)\Big\}\\
&\le a_B.
\end{aligned}
$$

(iii) For $i > 1$ using the induction hypothesis gives

$$
\begin{aligned}
v_n^{(\alpha)}(\mathbf{r} + \mathbf{e}_i) &= \min\Big\{r_0 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}((r_1, \ldots, r_{D-1}, k) + \mathbf{e}_{i-1}), h_{n,1}^{(\alpha)}\Big\}\\
&\le \min\Big\{r_0 + \alpha \sum_{k=0}^{\infty} q_k \Big(1 + v_{n-1}^{(\alpha)}(r_1, \ldots, r_{D-1}, k)\Big), h_{n,1}^{(\alpha)}\Big\}\\
&\le \min\Big\{r_0 + 1 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}(r_1, \ldots, r_{D-1}, k), h_{n,1}^{(\alpha)}\Big\}\\
&\le 1 + v_n^{(\alpha)}(\mathbf{r}).
\end{aligned}
$$

For $i = 1$ we have that

$$\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_1) &= \min\Big\{r_0 + 1 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}(r_1,\ldots,r_{D-1},k), h_{n,1}^{(\alpha)}\Big\} \\
&\leq 1 + v_n^{(\alpha)}(\mathbf{r}).
\end{aligned}$$

(iv) For $j > i > 1$ using the induction hypothesis gives

$$\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_j) &= \min\Big\{r_0 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}((r_1,\ldots,r_{D-1},k)+\mathbf{e}_{j-1}), h_{n,1}^{(\alpha)}\Big\} \\
&\geq \min\Big\{r_0 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}((r_1,\ldots,r_{D-1},k)+\mathbf{e}_{i-1}), h_{n,1}^{(\alpha)}\Big\} \\
&= v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i).
\end{aligned}$$

For $j > i = 1$ we use Theorem 2.3(iii) to obtain

$$\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_j) &= \min\Big\{r_0 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}((r_1,\ldots,r_{D-1},k)+\mathbf{e}_{j-1}), h_{n,1}^{(\alpha)}\Big\} \\
&\leq \min\Big\{r_0 + \alpha \sum_{k=0}^{\infty} q_k \Big(1 + v_{n-1}^{(\alpha)}(r_1,\ldots,r_{D-1},k)\Big), h_{n,1}^{(\alpha)}\Big\} \\
&\leq \min\Big\{r_0 + 1 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}^{(\alpha)}(r_1,\ldots,r_{D-1},k), h_{n,1}^{(\alpha)}\Big\} \\
&\leq v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_1).
\end{aligned}$$

(v) For $k = 1$ this is equivalent to (iii). Next suppose that (iv) holds for $k-1$, then it follows from Theorem 2.3(iii) that

$$v_n^{(\alpha)}(\mathbf{r}) \leq v_n^{(\alpha)}(\mathbf{r}+(k-1)(\mathbf{e}_i-\mathbf{e}_j)) \leq v_n^{(\alpha)}(\mathbf{r}+k(\mathbf{e}_i-\mathbf{e}_j)).$$

(vi) Repeated application of Theorem 2.3(v), while using the fact that $r_{0,i} \leq r'_{0,i}$ ($i = 0,\ldots,D-2$), yields

$$\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}) &\leq v_n^{(\alpha)}(r'_0, r'_0 + r'_1 - r_0, r_2, \ldots, r_{D-1}) \\
&\leq v_n^{(\alpha)}(r'_0, \ldots, r'_{i-1}, r'_{0,i} - r_{0,i-1}, r_{i+1}, \ldots, r_{D-1}) \quad (i = 2,\ldots,D-2) \\
&\leq v_n^{(\alpha)}(r'_0, \ldots, r'_{D-2}, r'_{0,D-1} - r_{0,D-2}) \\
&= v_n^{(\alpha)}(\mathbf{r}'),
\end{aligned}$$

where the last equality uses $r_{0,D-1} = r'_{0,D-1}$. □

Proof of Theorem 2.5.
(i) First suppose that $\pi_n^*(\mathbf{r}+\mathbf{e}_i) = 1$. Then $\pi_n^*(\mathbf{r}+\mathbf{e}_i+\mathbf{e}_j) = 1$ by Theorem 2.4(ii), and hence

$$v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i+\mathbf{e}_j) - v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i) = h_{n,1}^{(\alpha)} - h_{n,1}^{(\alpha)} = 0 \leq v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_j) - v_n^{(\alpha)}(\mathbf{r}),$$

where the last inequality follows from Theorem 2.3(i).

Next suppose that $\pi_n^*(\mathbf{r}+\mathbf{e}_i)=0$ and $\pi_n^*(\mathbf{r}+\mathbf{e}_j)=1$ (by Theorem 2.4(iii) this is only possible if $i>j$). Then again $\pi_n^*(\mathbf{r}+\mathbf{e}_i+\mathbf{e}_j)=1$ by Theorem 2.4(ii), and hence

$$v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i+\mathbf{e}_j)-v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i)=h_{n,1}^{(\alpha)}-v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i)\le h_{n,1}^{(\alpha)}-v_n^{(\alpha)}(\mathbf{r})=v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_j)-v_n^{(\alpha)}(\mathbf{r}),$$

again using Theorem 2.3(i).

Finally, suppose that $\pi_n^*(\mathbf{r}+\mathbf{e}_i)=\pi_n^*(\mathbf{r}+\mathbf{e}_j)=0$. If $i,j>1$ then

$$\begin{aligned}
& v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i+\mathbf{e}_j)-v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i)\\
\le\ & h_{n,0}^{(\alpha)}(\mathbf{r}+\mathbf{e}_i+\mathbf{e}_j)-h_{n,0}^{(\alpha)}(\mathbf{r}+\mathbf{e}_i)\\
=\ & \alpha\sum_{k=0}^{\infty}q_k\Big(v_{n-1}^{(\alpha)}((r_1,\dots,r_{D-1},k)+\mathbf{e}_{i-1}+\mathbf{e}_{j-1})-v_{n-1}^{(\alpha)}((r_1,\dots,r_{D-1},k)+\mathbf{e}_{i-1})\Big)\\
\le\ & \alpha\sum_{k=0}^{\infty}q_k\Big(v_{n-1}^{(\alpha)}((r_1,\dots,r_{D-1},k)+\mathbf{e}_{j-1})-v_{n-1}^{(\alpha)}(r_1,\dots,r_{D-1},k)\Big)\\
=\ & h_{n,0}^{(\alpha)}(\mathbf{r}+\mathbf{e}_j)-h_{n,0}^{(\alpha)}(\mathbf{r})\\
=\ & v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_j)-v_n^{(\alpha)}(\mathbf{r}),
\end{aligned}$$

where the second inequality follows from the induction hypothesis. In the last equality note that $v_n^{(\alpha)}(\mathbf{r})=h_{n,0}^{(\alpha)}(\mathbf{r})$ since $\pi_n^*(\mathbf{r})=0$ by Theorem 2.4(ii). If $i=1$ then

$$\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_1+\mathbf{e}_j)-v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_1) &\le h_{n,0}^{(\alpha)}(\mathbf{r}+\mathbf{e}_1+\mathbf{e}_j)-h_{n,0}^{(\alpha)}(\mathbf{r}+\mathbf{e}_1)\\
&= h_{n,0}^{(\alpha)}(\mathbf{r}+\mathbf{e}_j)-h_{n,0}^{(\alpha)}(\mathbf{r})=v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_j)-v_n^{(\alpha)}(\mathbf{r}),
\end{aligned}$$

while if $j=1$ then

$$\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i+\mathbf{e}_1)-v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_i) &\le h_{n,0}^{(\alpha)}(\mathbf{r}+\mathbf{e}_i+\mathbf{e}_1)-h_{n,0}^{(\alpha)}(\mathbf{r}+\mathbf{e}_i)=1\\
&= h_{n,0}^{(\alpha)}(\mathbf{r}+\mathbf{e}_1)-h_{n,0}^{(\alpha)}(\mathbf{r})=v_n^{(\alpha)}(\mathbf{r}+\mathbf{e}_1)-v_n^{(\alpha)}(\mathbf{r}).
\end{aligned}$$

(ii) First, if $\pi_n^*(\mathbf{r}-\mathbf{e}_i+\mathbf{e}_{i-k})=1$ then obviously

$$v_n^{(\alpha)}(\mathbf{r}-\mathbf{e}_j+\mathbf{e}_{j-k})\le h_{n,1}^{(\alpha)}=v_n^{(\alpha)}(\mathbf{r}-\mathbf{e}_i+\mathbf{e}_{i-k}).$$

Next, if $\pi_n^*(x-\mathbf{e}_i+\mathbf{e}_{i-k})=0$ and $j-k>i-k>1$ then the result immediately follows from the induction hypothesis. Finally, suppose that $\pi_n^*(x-\mathbf{e}_i+\mathbf{e}_{i-k})=0$ and $i-k=1$. Then we also need the fact that

$$\begin{aligned}
v_{n-1}^{(\alpha)}((r_1,\dots,r_{D-1},k)-\mathbf{e}_j+\mathbf{e}_{j-k}) &\le v_{n-1}^{(\alpha)}((r_1,\dots,r_{D-1})-\mathbf{e}_i+\mathbf{e}_1)\\
&\le 1+v_n^{(\alpha)}((r_1,\dots,r_{D-1})-\mathbf{e}_i),
\end{aligned}\tag{2.90}$$

where the first inequality follows from the induction hypothesis and the second from Theorem 2.3(iii). Using (2.90) it follows that

$$\begin{aligned}
v_n^{(\alpha)}(\mathbf{r}-\mathbf{e}_j+\mathbf{e}_{j-k}) &\le r_0+\alpha\sum_{k=0}^{\infty}q_k v_{n-1}^{(\alpha)}((r_1,\dots,r_{D-1},k)-\mathbf{e}_{j-1}+\mathbf{e}_{j-k-1})\\
&\le r_0+1+\alpha\sum_{k=0}^{\infty}q_k v_{n-1}^{(\alpha)}((r_1,\dots,r_{D-1},k)-\mathbf{e}_{i-1})\\
&= v_n^{(\alpha)}(\mathbf{r}-\mathbf{e}_i+\mathbf{e}_{i-k}).\quad\square
\end{aligned}$$

Chapter 3

The continuous-time service model

3.1 Introduction

In the previous chapter we have studied a discrete-time formulation of the service model where a batch service can only be started at the end of a period. We found that for a delay-limit of D periods a complete state description is provided by the D-dimensional vector $(r_0, \ldots, r_{D-1})$ with r_i the number of customers with a residual delay-limit of i periods $(i = 0, \ldots, D-1)$. Therefore the optimal policy for this model can be computed by means of a MDP with a D-dimensional state space, but due to the curse of dimensionality solution of the model becomes computationally infeasible already for moderate values of D. In this chapter we consider a continuous-time formulation of the service model where the delay-limit equals D time units and a batch service can be started at any time. Whereas for the discrete-time model only the probability distribution of the total demand per period $\{q_k;\ k = 0, 1, \ldots\}$ is needed, for the continuous-time model we need to be more specific about the arrival process. We will assume throughout this chapter that customer arrivals follow a Poisson process $\{N(t), t \geq 0\}$ with parameter λ. Let A_i $(i = 1, 2, \ldots)$ be the interarrival time between the $(i-1)^{\text{th}}$ and i^{th} arrival, and $B_i := \sum_{j=1}^{i} A_j$ $(i = 1, 2, \ldots)$ the arrival epoch of the i^{th} customer. Obviously, $\{A_i\}$ is a sequence of i.i.d. exponential random variables with mean $\frac{1}{\lambda}$, and B_i has an Erlang-i distribution with mean $\frac{i}{\lambda}$.

Unfortunately, if the delay-limit is constant and equal to D time units, then an optimal policy for the resulting model is extremely complex. The reason for this is that a complete state description for this model must include the delay (or the residual delay-limit) of every waiting customer and hence is of infinite dimension. Specifically, a complete state description is provided by

$$(n; d_1, \ldots, d_n) \quad (n \in I\!N;\ 0 < d_1 < \cdots < d_n \leq D), \tag{3.1}$$

with n the number of waiting customers and d_i $(i = 1, \ldots, n)$ the residual delay-limit of the customer with the i^{th} longest waiting time. This would lead to a state space

$$\bigcup_{n=0}^{\infty} \Big\{(d_1, \ldots, d_n) \mid 0 < d_1 < \cdots < d_n \leq D\Big\}. \tag{3.2}$$

The discrete-time model of the previous chapter can be seen as a discretization of the continuous-time model. Suppose that the delay-limit D is divided into D' periods of $\frac{D}{D'}$

time units and that a batch service can only be started at the end of a period. Then we can group all waiting customers into D' groups according to

$$r_n := \left|\left\{i \mid n\frac{D}{D'} < d_i \leq (n+1)\frac{D}{D'}\right\}\right| \quad (n = 0, \ldots, D'-1), \tag{3.3}$$

so that r_n denotes the number of customers that must be served within n periods. Also, we can group all arriving customers in a period into one group, since only the total demand per period is relevant. If the arrival process is a Poisson process with parameter λ, then the total demand per period has a Poisson distribution with parameter $\frac{\lambda D}{D'}$. In this way we can transform the continuous-time model into the discrete-time model, for which it is not necessary to keep track of the residual delay-limit of every individual customer. We can apply all results of the previous chapter by replacing D with D'. Moreover, referring to the discussion of section 2.8, the new delay-limit of D' periods can be treated as a decision variable by adjusting the length of a period accordingly. The choice of D' should be based on the trade-off between accuracy and complexity of the resulting model.

Since the optimal policy for the continuous-time model is infeasible and of no practical use, we will restrict attention to heuristic policies and approximations. In the next section we introduce the so-called Generalized Critical-Group (GCG) policy, which can be seen as the continuous-time analogon of the Critical-Group policy (see section 2.3). Under a GCG-policy with parameters C and K a batch service is started as soon as the total number of waiting customers with a residual delay-limit of at most C time units ($0 \leq C \leq D$) reaches a control-limit K. The class of GCG-policies also includes the continuous-time analogon of the Total-Demand policy (see section 2.4), by setting $C = D$. It turns out that the analysis of a regenerative cycle of the GCG-policy leads to a M/D/∞ queueing system with Poisson arrivals, constant service times of C time units and ample service capacity. Specifically, the problem of finding the cycle length is equivalent to finding the time until the number of customers in a M/D/∞ queue with service time C reaches the level K. Unfortunately, this is a difficult problem and there is no simple expression for the distribution or the expectation of these "first entrance times". In section 3.3 we will derive a closed-form expression for the distribution function, which is not suited for numerical purposes but can be used as the basis for an accurate approximation of the expected first entrance times.

Under the GCG-policy an epoch at which a batch service is started does not necessarily coincide with the expiration of a delay-limit, and hence the GCG-policy can be further improved by postponing the batch service until the next delay-limit expires. This is the idea of the Improved Generalized Critical-Group (IGCG) policy of section 3.4: start a batch service at the first epoch that a delay-limit expires after the moment at which the number of waiting customers with a residual delay-limit of at most C time units ($0 \leq C \leq D$) has reached the level K. Unfortunately, using a similar approach as for the GCG-policy leads to complications, caused by the fact that the cycle length is not a stopping time for $\{N(t)\}$. Therefore we use an alternative approach in the form of an approximate embedded Markov chain.

In section 3.5 we turn to a model where the delay-limit is not constant but has an Erlang-n distribution, i.e., the delay-limit consists of n exponentially distributed phases. This "phase-type model" is similar to the discrete-time model, the most important difference being that the period length is exponentially distributed instead of constant (in the discrete-time model the delay-limit consists of D constant phases, whereas in the phase-type model the delay-limit consists of n exponentially distributed phases). This model can be used as an approximation for the continuous-time model (with D deterministic) by using an exponential phase distribution with mean $\frac{D}{n}$. Since the Erlang-n distribution with mean D converges to a constant D as n tends to infinity, it is intuitively clear that the phase-type model "converges" to the continuous-time model where the accuracy of the approximation increases with n. However, the computational burden also increases with n and, just as for the method of discretization, there is a trade-off between accuracy and efficiency. We present a complete Markov chain analysis of the special case $n = 1$, corresponding to an exponentially distributed delay-limit.

We conclude this chapter in section 3.6 with a numerical comparison of the various policies of this chapter. By using the same parameter settings as in section 2.7 we are also able to compare the continuous-time policies with the discrete-time policies of the previous chapter.

3.2 The Generalized Critical-Group policy

In this section we focus on a broad class of (heuristic) policies for the continuous-time service model, namely the class of Generalized Critical-Group (GCG) policies. Under a GCG-policy with parameters C and K a batch service is started as soon as the total number of waiting customers with a residual delay-limit of at most C time units ($0 \leq C \leq D$) reaches a control-limit K. The idea of this policy is similar to the Critical-Group policy for the discrete-time model: wait until more than K customers arrive within an interval of length C and start a batch service $D - C$ time units later, to ensure that this group is included in the batch. Whereas for the discrete-time CG-policy $C = 1$ (the length of one period), for the continuous-time GCG-policy it is not clear which value of C to use. Instead C can be treated as a decision variable, giving the policy additional flexibility. In fact, the class of GCG-policies also includes the continuous-time analogon of the TD-policy (set $C = D$), and thus generalizes both the CG-policy and the TD-policy.

We start with some general results for a given policy π. Define

$$X_\pi(t) \ := \ \text{number of waiting customers at time } t \text{ under policy } \pi \quad (t \geq 0),$$

and let $X_\pi(0) := 0$. Since a batch service clears the system and due to the assumption of Poisson arrivals, $\{X_\pi(t)\}$ is a regenerative process for any policy π, with the epochs at which a batch service is started as regeneration epochs (provided that the expected time between two consecutive batch services is finite). Therefore, just as for all policies in the previous chapter, we can restrict the analysis to a single regenerative cycle. Defining

$$\begin{aligned}
g_\pi &:= \text{expected average costs under policy } \pi; \\
S_\pi &:= \text{length of a cycle under policy } \pi; \\
Y_\pi &:= \text{number of individual services in a cycle under policy } \pi; \\
Z_\pi &:= \text{number of customers served through a batch service in a cycle under policy } \pi,
\end{aligned}$$

we have by the Renewal Reward Theorem that

$$g_\pi = \frac{a_B + b_B E\{Z_\pi\} + b_I E\{Y_\pi\}}{E\{S_\pi\}} \tag{3.4}$$

(see also section 2.2).

Before we turn to the GCG-policy, we consider the two extreme policies: the continuous-time Never-Batch (NB) and Only-Batch (OB) policies. Under the NB-policy all customers are given individual service when their delay-limit expires, and it is easily seen that

$$g_{\text{NB}} = \lambda b_I. \tag{3.5}$$

Note that the epochs at which a batch service is started cannot be used as regeneration epochs for $\{X_{\text{NB}}(t)\}$, since no batch services are done at all. Still, $\{X_{\text{NB}}(t)\}$ is a regenerative process by using the epochs at which $X_{\text{NB}}(t)$ becomes equal to zero as regeneration epochs. Moreover, since any customer receives an individual service D time units after arriving,

$$X_{\text{NB}}(t) = \begin{cases} N(t) & \text{if } 0 \le t \le D; \\ N(t-D,t) & \text{if } t > D, \end{cases} \tag{3.6}$$

where $N(t,u) := N(u) - N(t)$ $(u > t)$. Consequently, $X_{\text{NB}}(t)$ corresponds to the number of customers in the system at time t in a M/D/∞ queue (Poisson arrivals, deterministic service times of D time units and ample service capacity). This is an important observation that we will exploit in the analysis of the GCG-policy.

Under the OB-policy, all customers are served through a batch service and a batch service is started when the delay-limit of the first customer expires. As the first customer arrives at time A_1, the batch service is started at time $A_1 + D$, upon expiration of the delay-limit of this customer, with all customers that have arrived in the meantime being included in the batch. It follows that

$$E\{Y_{\text{OB}}\} = 0; \tag{3.7}$$

$$E\{Z_{\text{OB}}\} = E\{N(A_1 + D)\} = 1 + \lambda D; \tag{3.8}$$

$$E\{S_{\text{OB}}\} = E\{A_1 + D\} = \frac{1}{\lambda} + D, \tag{3.9}$$

and hence

$$g_{\text{OB}} = \frac{a_B + b_B E\{Z_{\text{OB}}\}}{E\{S_{\text{OB}}\}} = \lambda\left(b_B + \frac{a_B}{1+\lambda D}\right). \tag{3.10}$$

The OB-policy is a reasonable policy if the individual service cost (b_I) is much higher than the variable batch service cost (b_B), while the NB-policy will perform well if b_I is only slightly higher than b_B. In the intermediate region it is better to use a more advanced policy like the GCG-policy.

Now we turn to the analysis of the Generalized Critical-Group (GCG) policy, starting with the case $C = D$: start a batch service as soon as the total number of waiting customers reaches the level K (the continuous-time analogon of the Total-Demand policy). Since only individual services take place before the batch service is started, the cycle length for the continuous-time TD-policy is

$$S_{\text{TD}} = \min\{t \ge D : X_{\text{NB}}(t) = K\}, \tag{3.11}$$

with $X_{\text{NB}}(t)$ given by (3.6). As noted earlier, $\{X_{\text{NB}}(t)\}$ describes the number of customers over time in a M/D/∞ queue with service times D. It follows from (3.11) that S_{TD} corresponds to the first time that the number of customers in a M/D/∞ queue reaches the level K. In the following we will show that the expected average costs of the GCG-policy with parameters K and C can be expressed in terms of the expected first entrance time into level K of a M/D/∞ queue with service times C.

To this end, consider a M/D/∞ queue with a Poisson arrival process $\{N(t)\}$ and service times $C > 0$, and define

$$X^{(C)}(t) \quad := \quad \text{number of customers in the system at time } t \quad (t \geq 0);$$

(the dependence on C is needed later). In a M/D/∞ queue with constant service times C, any customer leaves the system exactly C time units after arriving. Therefore the number of customers present at a given time is just the number of customers that has arrived during the last C time units, i.e.,

$$X^{(C)}(t) = \begin{cases} N(t) & \text{if } 0 \leq t \leq C; \\ N(t-C,t) & \text{if } t > C \end{cases} \tag{3.12}$$

(see also (3.6)). Since $N(t,u)$ is Poisson($\lambda(u-t)$) distributed, it follows immediately from (3.12) that $X^{(C)}(t)$ has a Poisson distribution with mean λt if $t < C$ and mean λC if $t \geq C$, or

$$\Pr\{X^{(C)}(t) = k\} = e^{-\lambda \min\{t,C\}} \frac{(\lambda \min\{t,C\})^k}{k!} \quad (t \geq 0). \tag{3.13}$$

Next define

$$T_K^{(C)} \quad := \quad \min\{t \geq 0 : X^{(C)}(t) = K\} \quad (K = 1,2,\ldots); \tag{3.14}$$
$$N_K^{(C)} \quad := \quad \min\{n : X^{(C)}(B_n) = K\} \quad (K = 1,2,\ldots) \tag{3.15}$$

(where $X^{(C)}(B_n)$ includes the n^{th} customer), so that $T_K^{(C)}$ is the first entrance time into level K and $N_K^{(C)}$ is the index of the first customer that increases the level to K. Note that $T_K^{(C)}$ is a customer arrival epoch by definition, and that $N_K^{(C)} = N(T_K^{(C)})$. Now the number of customers in the system just after arrival of the i^{th} customer is below the level K if and only the $(i-K+1)^{\text{th}}$ customer has already left, i.e.,

$$\{N(B_i) < K\}\} \iff \{B_i > B_{i-K+1} + C\} \quad (i \geq K). \tag{3.16}$$

It follows from (3.16) that $\Pr\{N_K^{(C)} = k\} = 0$ for $k < K$ and

$$\begin{aligned} \Pr\{N_K^{(C)} = k\} &= \Pr\{T_K^{(C)} = B_k\} \\ &= \Pr\{B_i > B_{i-K+1} + C,\ i = K,\ldots,k-1;\ B_k \leq B_{k-K+1} + C\} \\ &= \Pr\{A_{i-K+2,i} > C,\ i = K,\ldots,k-1; A_{k-K+2,k} \leq C\} \end{aligned} \tag{3.17}$$

for $k \geq K$, where $A_{ij} := \sum_{k=i}^{j} A_k$. Since by (3.17) the event $\{N_K^{(C)} = k\}$ is independent of $\{A_{k+1}, A_{k+2}, \ldots\}$, $N_K^{(C)}$ is a stopping time for the sequence $\{A_i\}$ for any K. Therefore we can apply Wald's equation to obtain

$$E\{T_K^{(C)}\} = E\{B_{N_K^{(C)}}\} = E\Big\{\sum_{i=1}^{N_K^{(C)}} A_i\Big\} = E\{N_K^{(C)}\}E\{A_1\} = \frac{1}{\lambda}E\{N_K^{(C)}\}. \tag{3.18}$$

Remark. Relation (3.18) is a special case of the following more general result: Let $\{N(t)\}$ be a Poisson process with parameter λ, $M(t) := E\{N(t)\}$ and T a stopping time for $\{N(t)\}$, i.e., for any t the event $\{T \leq t\}$ only depends on $\{N(u), 0 \leq u \leq t\}$. Then

$$E\{N(T)\} = M(E\{T\}) = \lambda E\{T\}. \tag{3.19}$$

Whereas the distribution of $X^{(C)}(t)$ is trivial (see (3.13)), the distributions of $T_K^{(C)}$ and $N_K^{(C)}$ are extremely difficult. The main reason is that these random variables involve the transient behaviour of the process $\{X^{(C)}(t)\}$, which is not a Markov chain. In the next section we will study the distribution of $T_K^{(C)}$ in depth and derive an exact (but cumbersome) expression for $E\{T_K^{(C)}\}$. Since this expression is not suited for computational purposes, we also develop an approximation.

We are now ready to express g_{TD} and g_{GCG} in terms of $E\{T_K^{(D)}\}$ and $E\{T_K^{(C)}\}$, respectively. We have already found that

$$E\{S_{\text{TD}}\} = E\{T_K^{(D)}\}, \tag{3.20}$$

and it remains to find expressions for $E\{Y_{\text{TD}}\}$ and $E\{Z_{\text{TD}}\}$. Obviously, since a batch service is started when exactly K customers are present,

$$E\{Z_{\text{TD}}\} = K. \tag{3.21}$$

Individual services can only occur if $T_K^{(D)} > D$; in this case the last K of the total of $N_K^{(D)}$ customers are included in the batch, whence the first $N_K^{(D)} - K$ customers are served individually. Using (3.18) it follows that

$$E\{Y_{\text{TD}}\} = E\{N_K^{(D)} - K\} = \lambda E\{T_K^{(D)}\} - K. \tag{3.22}$$

Substituting (3.20)–(3.22) into (3.4) we find the following expression for the expected average costs of the TD-policy as a function of the control-limit K:

$$g_{\text{TD}}(K) = \frac{a_B + b_B K + b_I\left(\lambda E\{T_K^{(D)}\} - K\right)}{E\{T_K^{(D)}\}} = \lambda b_I - \frac{(b_I - b_B)K - a_B}{E\{T_K^{(D)}\}}. \tag{3.23}$$

Hence the computation of $g_{\text{TD}}(K)$ reduces to the computation of the expected first entrance times $E\{T_K^{(D)}\}$.

It turns out that similar results hold for the GCG-policy with parameters C and K. Defining the critical group as the customers with a residual delay-limit of C time units or less, the GCG-policy prescribes to start a batch service as soon as the size of the critical group reaches the level K (this justifies the name of the policy). Since the critical group at time t consists of the customers that arrived in the interval $[t-D, t-D+C]$, we have that

$$\begin{aligned} S_{\text{GCG}} &= \min\Big\{t : N(t-D, t-D+C) = K\Big\} \\ &= \min\Big\{u \geq C : N(u-C, u) = K\Big\} + D - C \\ &= T_K^{(C)} + D - C, \end{aligned} \tag{3.24}$$

and

$$E\{S_{\text{GCG}}\} = E\{T_K^{(C)}\} + D - C. \tag{3.25}$$

Alternatively, observe that the first time that K customers arrive within a timespan of C time units corresponds to $T_K^{(C)}$, and that the batch service is started $D - C$ time units later. The batch consists of the critical group of K customers plus the customers that have arrived during the last $D - C$ time units, implying that

$$E\{Z_{\text{GCG}}\} = K + E\Big\{N(T_K^{(C)}, T_K^{(C)} + D - C)\Big\} = K + \lambda(D - C). \tag{3.26}$$

Also, since $N_K^{(C)}$ is the index of the last customer in the critical group, $N_K^{(C)} - K$ customers receive an individual service. This gives

$$E\{Y_{\text{GCG}}\} = E\{N_K^{(C)} - K\} = \lambda E\{T_K^{(C)}\} - K. \tag{3.27}$$

Substituting (3.25)-(3.27) into (3.4) we find the following expression for the expected average costs of the GCG-policy as a function of the parameters C and K:

$$\begin{aligned} g_{\text{GCG}}(C, K) &= \frac{a_B + b_B\big(K + \lambda(D - C)\big) + b_I\big(\lambda E\{T_K^{(C)}\} - K\big)}{E\{T_K^{(C)}\} + D - C} \\ &= \lambda b_I - \frac{(b_I - b_B)\big(K + \lambda(D - C)\big) - a_B}{E\{T_K^{(C)}\} + D - C}. \end{aligned} \tag{3.28}$$

Note that (3.28) reduces to (3.23) by setting $C = D$. It remains to compute the expected first entrance time $E\{T_K^{(C)}\}$, and in the next section we will focus on this problem.

We conclude this section with a straightforward modification of the GCG-policy to incorporate the fact that a batch service should not be started within D time units of the previous one, as no individual services are needed during this time (see also Corollary 2.1 in section 2.6). This is achieved by adjusting the cycle length such that a batch service is never started before time D, i.e.,

$$S'_{\text{GCG}} = \max\{D, T_K^{(C)} + D - C\}. \tag{3.29}$$

It follows that

$$\begin{aligned} E\{S'_{\text{GCG}}\} &= \int_{t=0}^{\infty} \Pr\{S'_{\text{GCG}} > t\}\, dt \\ &= D + \int_{t=D}^{\infty} \Pr\{T_K^{(C)} + D - C > t\}\, dt \\ &= D + \int_{t=C}^{\infty} \Pr\{T_K^{(C)} > t\}\, dt, \end{aligned} \tag{3.30}$$

for which we need the distribution function of $T_K^{(C)}$. It is easily seen that the number of individual services in a cycle remains the same, i.e.,

$$E\{Y'_{\text{GCG}}\} = E\{N_K^{(C)} - K\} = \lambda E\{T_K^{(C)}\} - K. \tag{3.31}$$

On the other hand, the size of the critical group may now exceed K if $S'_{\text{GCG}} = D$. Conditioning on $N(C)$ we find that

$$\begin{aligned} E\{Z'_{\text{GCG}}\} &= K\sum_{k=0}^{K-1} e^{-\lambda C}\frac{(\lambda C)^k}{k!} + \sum_{k=K}^{\infty} ke^{-\lambda C}\frac{(\lambda C)^k}{k!} + \lambda(D-C) \\ &= \lambda D + \sum_{k=0}^{K-1}(K-k)e^{-\lambda C}\frac{(\lambda C)^k}{k!}. \end{aligned} \tag{3.32}$$

Substituting (3.30)–(3.32) into (3.4) we obtain the expected average costs for the modified GCG-policy. The resulting expression can no longer be simplified as in (3.23) and (3.28), because it does not only depend on the mean but also on the distribution of $T_K^{(C)}$.

3.3 On the first entrance times of a M/D/∞ queue

The purpose of this section is to find the distribution and the mean of $T_K^{(C)}$ (the first entrance time of a M/D/∞ queue) that will enable us to compute the expected average costs of the TD-policy in (3.23) and the GCG-policy in (3.28). Although a reasonable amount of research has been reported on the transient behaviour of a M/G/∞ or M/D/∞ queue (see e.g. [Gross&Harris 1985], 5.2.3; [Takács 1962], Chapter 3), this particular problem seems to have received no attention. The problem of finding first passage times is extensively studied, but only for Markov chains and diffusion processes. As the process $\{X^{(C)}(t)\}$ is not a Markov chain, these techniques are not applicable. It also explains the intrinsic difficulty of the problem. We note that the content of this section is rather technical, and not essential for the remainder of the thesis; it is based on [Jansen 1996].

3.3.1 Preliminary results

Define

$$F_{T_K^{(C)}}(t) := \Pr\{T_K^{(C)} \le t\}, \quad \bar{F}_{T_K^{(C)}}(t) := 1 - F_{T_K^{(C)}}(t) \quad (t \ge 0).$$

First note that $T_1^{(C)} = A_1$, and hence

$$F_{T_1^{(C)}}(t) = 1 - e^{-\lambda t} \quad (t \ge 0), \quad E\{T_1^{(C)}\} = \frac{1}{\lambda}. \tag{3.33}$$

Clearly, $\Pr\{T_K^{(C)} > t \mid N(t) < K\} = 1$. By (3.16) we have that

$$\begin{aligned} \Pr\{T_K^{(C)} > t \mid N(t) \ge K\} &= \Pr\{X^{(C)}(s) < K,\ 0 \le s \le t\} \\ &= \Pr\{X^{(C)}(B_i) < K,\ i = 1, \dots, N(t)\} \\ &= \Pr\{B_K > B_1 + C, \dots, B_{N(t)} > B_{N(t)-K+1} + C\} \end{aligned} \tag{3.34}$$

for any $t \ge 0$. Moreover,

$$\bar{F}_{T_K^{(C)}}(t) = \Pr\{N(t) < K\} = \sum_{k=0}^{K-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!} \quad (0 \le t \le C). \tag{3.35}$$

Conditioning on $\{N(t) = k\}$ and $B_1, \ldots, B_k$, and using the fact that

$$f_{B_1,\ldots,B_k|N(t)=k}(t_1,\ldots,t_k) = \frac{k!}{t^k} \quad (0 \le t_1 \le \cdots \le t_k \le t), \tag{3.36}$$

it follows from (3.34) that

$$\begin{aligned}\bar{F}_{T_K^{(C)}}(t) &= \sum_{k=0}^{K-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!} + \sum_{k=K}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!} \int\limits_{\substack{t_1,\ldots,t_k:\\ 0\le t_1\le\cdots\le t_k\le t}} \frac{k!}{t^k}\cdot \\ &\cdot \Pr\{B_i > B_{i-K+1} + C,\ i = K,\ldots,k \mid N(t) = k;\ B_1 = t_1,\ldots,B_k = t_k\}\, dt_k \cdots dt_1 \\ &= \sum_{k=0}^{K-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!} + \sum_{k=K}^{\lceil \frac{t}{C}\rceil (K-1)} e^{-\lambda t}\lambda^k \int\limits_{\substack{t_1,\ldots,t_k:\\ 0\le t_1\le\cdots\le t_k\le t;\\ t_i > t_{i-K+1}+C, i=K,\ldots,k}} 1\, dt_k \cdots t_1 \quad (t > C).\end{aligned} \tag{3.37}$$

Note that the maximal number of arrivals in $[0,t]$ for which probability (3.34) is nonzero consists of K–1 arrivals in each of the intervals $[(i-1)C, iC)$ $(i = 1,\ldots,\lfloor\frac{t}{C}\rfloor)$ and $[\lfloor\frac{t}{C}\rfloor C, t)$, or a total of $\lceil\frac{t}{C}\rceil(K-1)$ arrivals. It turns out that the integral in (3.37) is extremely difficult to elaborate (if at all possible), except for $K = 2$. Therefore we will use an alternative combinatorial approach in the next subsection.

For $K = 2$ it is possible to simplify (3.37), but we need the following result.

Lemma 3.1 *For any $k = 1, 2, \ldots$ and $j = 0, 1, \ldots, k-1$ we have that*

$$I(j,k) := \int\limits_{\substack{t_{j+1},\ldots,t_k:\\ t_{i-1}+C\le t_i\le t-(k-i)C, i=j+1,\ldots,k}} dt_k \cdots dt_{j+1} = \frac{\left(t - t_j - (k-j)C\right)^{k-j}}{(k-j)!}. \tag{3.38}$$

Proof. We use backward induction on j with k fixed. For $j = k-1$ (3.38) trivially holds. Suppose that (3.38) holds for $j = j'+1$. It follows that

$$\begin{aligned} I(j',k) &= \int_{t_{j'+1}=t_{j'}+C}^{t-(k-j'-1)C} I(j'+1,k)\, dt_{j'+1} \\ &= \int_{t_{j'+1}=t_{j'}+C}^{t-(k-j'-1)C} \frac{(t - t_{j'+1} - (k-j'-1)C)^{k-j'-1}}{(k-j'-1)!}\, dt_{j'+1} \\ &= \int_{t_{j'+1}=0}^{t-t_{j'}-(k-j')C} \frac{t_{j'+1}^{k-j'-1}}{(k-j'-1)!}\, dt_{j'+1} \\ &= \frac{(t - t_{j'} - (k-j')C)^{k-j'}}{(k-j')!}, \end{aligned}$$

and hence (3.38) holds for $j = j'$. □

Theorem 3.1 *(i) The distribution of $T_2^{(C)}$ is given by*

$$\bar{F}_{T_2^{(C)}}(t) = \sum_{k=0}^{\lceil \frac{t}{C} \rceil} e^{-\lambda t} \frac{\left(\lambda(t-(k-1)C)\right)^k}{k!} \quad (t \geq 0). \tag{3.39}$$

(ii) The mean of $T_2^{(C)}$ is given by

$$E\{T_2^{(C)}\} = \frac{1}{\lambda} \frac{2 - e^{-\lambda C}}{1 - e^{-\lambda C}}. \tag{3.40}$$

Proof. It follows from Lemma 3.1 with $j = 1$ that

$$\begin{aligned}
\int\limits_{\substack{t_1,\ldots,t_k: \\ 0 \leq t_1 \leq \cdots \leq t_k \leq t; \\ t_i > t_{i-1}+C, i=2,\ldots,k}} 1 \, dt_k \cdots t_1 &= \int\limits_{\substack{t_1,\ldots,t_k: \\ 0 \leq t_1 \leq t-(k-1)C; \\ t_{i-1}+C \leq t_i \leq t-(k-i)C, i=2,\ldots,k}} 1 \, dt_k \cdots dt_1 \\
&= \int\limits_{t_1=0}^{t-(k-1)C} \frac{\left(t - t_1 - (k-1)C\right)^{k-1}}{(k-1)!} \, dt_1 \\
&= \frac{\left(t-(k-1)C\right)^k}{k!} \quad (t \geq (k-1)C).
\end{aligned} \tag{3.41}$$

Substituting (3.41) into (3.37) yields (3.39).
(ii) Integrating (3.38) over t gives

$$\begin{aligned}
E\{T_2^{(C)}\} &= \int\limits_{t=0}^{\infty} \left(e^{-\lambda t} + \sum_{k=1}^{\lceil \frac{t}{C} \rceil} e^{-\lambda t} \frac{\left(\lambda(t-(k-1)C)\right)^k}{k!} \right) dt \\
&= \int\limits_{t=0}^{\infty} e^{-\lambda t} \, dt + \sum_{k=1}^{\infty} \int\limits_{t=(k-1)C}^{\infty} e^{-\lambda t} \frac{\left(\lambda(t-(k-1)C)\right)^k}{k!} \, dt \\
&= \frac{1}{\lambda} + \sum_{k=1}^{\infty} \int\limits_{u=0}^{\infty} e^{-\lambda(u+(k-1)C)} \frac{(\lambda u)^k}{k!} \, du \\
&= \frac{1}{\lambda} + \sum_{k=1}^{\infty} (e^{-\lambda C})^{k-1} \frac{1}{\lambda} \int\limits_{u=0}^{\infty} \frac{\lambda^{k+1} u^k e^{-\lambda u}}{k!} \, du \\
&= \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{k=1}^{\infty} (e^{-\lambda C})^{k-1} \\
&= \frac{1}{\lambda} \frac{2 - e^{-\lambda C}}{1 - e^{-\lambda C}}.
\end{aligned}$$

A second proof of (3.40) exploits relation (3.18). It follows from (3.17) that $N_2^{(C)} - 1$ is geometrically distributed with parameter $1 - e^{-\lambda C}$, so that $E\{N_2^{(C)} - 1\} = \frac{1}{1-e^{-\lambda C}}$.

Applying (3.18) then yields

$$E\{T_2^{(C)}\} = \frac{1}{\lambda}E\{N_2^{(C)}\} = \frac{1}{\lambda}\left(1 + \frac{1}{1-e^{-\lambda C}}\right) = \frac{1}{\lambda}\,\frac{2-e^{-\lambda C}}{1-e^{-\lambda C}}.$$

A third proof of (3.40) is by conditioning on the arrival epoch of the second customer, which gives

$$\begin{aligned} E\{T_2^{(C)}\} &= \frac{1}{\lambda} + \int_{t=0}^{C} t\lambda e^{-\lambda t}dt + e^{-\lambda C}\left(C + E\{T_2^{(C)}\}\right) \\ &= \frac{1}{\lambda} + \frac{1}{\lambda}(1-e^{-\lambda C}) + e^{-\lambda C}E\{T_2^{(C)}\}, \end{aligned} \tag{3.42}$$

and solving (3.42) for $E\{T_2^{(C)}\}$. □

Remark. As an anonymous referee pointed out, the results of Theorem 3.1 can also be derived by using the "method of collective marks" (for a description of this method see [Runnenburg 1965] or [Kleinrock 1975], Chapter 7). The idea is to interpret the Laplace transform of a random variable as a probability, so that it can be derived via probabilistic arguments. For $T_2^{(C)}$ this leads to

$$E\{e^{-sT_2^{(C)}}\} = \frac{\lambda}{s+\lambda}\,\frac{\lambda(1-e^{-(s+\lambda)D})}{s+\lambda(1-e^{-(s+\lambda)D})} \quad (s>0), \tag{3.43}$$

from which (3.39) and (3.40) follow.

3.3.2 A combinatorial approach

In this subsection we use a combinatorial approach to derive $\bar{F}_{T_K^{(C)}}(nC)$ $(n = 1,2,\ldots)$, the distribution function of $T_K^{(C)}$ at integer multiples of C. To this end we divide the time interval $[0, nC]$ into n periods of length C. Define

$$\begin{aligned} I_i &:= \text{time interval } [(i-1)C, iC) \quad (i = 1,2,\ldots); \\ N_i(t) &:= \text{number of arrivals in } [(i-1)C, (i-1)C+t) \\ &= N((i-1)C+t) - N((i-1)C) \quad (0 \le t < C;\ i = 1,\ldots,n); \\ \mathbf{N}(t) &:= \left(N_1(t),\ldots,N_n(t)\right) \quad (0 \le t < C); \\ \mathbf{k} &:= (k_1,\ldots,k_n); \\ R_{ij} &:= \inf\{0 \le t < C : N_i(t) = j\} \quad (i = 1,\ldots,n;\ j = 1,2,\ldots); \\ M(t) &:= N_1(t) + \ldots + N_n(t) \quad (0 \le t < C); \\ R_k &:= \inf\{0 \le t < C : M(t) = k\} \quad (k = 1,2,\ldots); \\ X_{ki} &:= N_i(R_k) \quad (k = 1,2,\ldots;\ i = 1,\ldots,n); \\ \mathbf{X}_k &:= (X_{k1},\ldots,X_{kn}) \quad (k = 1,2,\ldots); \\ \mathbf{x}_k &:= (x_{k1},\ldots,x_{kn}). \end{aligned}$$

Note that X_{ki} is the number of arrivals in I_i up to the time of the k^{th} arrival of $\{M(t)\}$, so that $\sum_{i=1}^{n} X_{ki} = k$. Conditioning on $\mathbf{N}(C)$ we have that

$$\bar{F}_{T_K^{(C)}}(nC) = \sum_{\substack{k_i=0,\ldots,K-1;\\ i=1,\ldots,n}} \left(\prod_{i=1}^{n} e^{-\lambda C}\frac{(\lambda C)^{k_i}}{k_i!}\right) \Pr\{X(t) < K,\ 0 \le t \le nC \mid \mathbf{N}(C) = \mathbf{k}\}. \quad (3.44)$$

Next observe that every arrival in I_{i-1} corresponds to a departure in I_i. Therefore,

$$\begin{aligned}
& \Pr\{X(t) < K,\ 0 \le t \le nC \mid \mathbf{N}(C) = \mathbf{k}\} \\
= {} & \Pr\{X((i-1)C + t) < K,\ 0 \le t \le C,\ i = 1,\ldots,n \mid \mathbf{N}(C) = \mathbf{k}\} \\
= {} & \Pr\{N_{i-1}(C) - N_{i-1}(t) + N_i(t) < K,\ 0 \le t \le C,\ i = 1,\ldots,n \mid \mathbf{N}(C) = \mathbf{k}\} \\
= {} & \Pr\{N_i(t) - N_{i-1}(t) < K - k_{i-1},\ 0 \le t \le C,\ i = 1,\ldots,n \mid \mathbf{N}(C) = \mathbf{k}\} \\
= {} & \Pr\{X_{ki} - X_{k,i-1} < K - k_{i-1},\ k = 1,\ldots,k_{1;n},\ i = 1,\ldots,n \mid \mathbf{N}(C) = \mathbf{k}\}, \quad (3.45)
\end{aligned}$$

with $N_0(t) = X_{k0} = k_0 := 0$ and $k_{m;n} := \sum_{i=m}^{n} k_i$. Probability (3.45) is completely determined by the sample paths of the n-dimensional finite discrete stochastic process $\{\mathbf{X}_k;\ k = 1,\ldots,k_{1;n}\}$. Moreover, it can be shown that every sample path of $\{\mathbf{X}_k\}$ has equal probability. This non-trivial result is stated formally in the next theorem, and proved in Appendix 3.A.

Theorem 3.2 *For $x_{ki} \le k_i$ $(i = 1,\ldots,n)$ with $\sum_{i=1}^{n} x_{ki} = k$ and $\mathbf{x}_k - \mathbf{x}_{k-1} \in \{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ $(k = 1,\ldots,k_{1;n})$, and any $t \ge 0$, we have that*

$$\Pr\{\mathbf{X}_k = \mathbf{x}_k;\ k = 1,\ldots,k_{1;n} \mid \mathbf{N}(t) = \mathbf{k}\} = \binom{k_{1,n}}{k_1,\ldots,k_n}^{-1}.$$

Proof. See Appendix 3.A.

Theorem 3.2 implies that probability (3.45) is equal to the number of paths from $(0,\ldots,0)$ to $(k_1,\ldots,k_n)$ that satisfy the conditions

$$x_i - x_{i-1} < K - k_{i-1} \quad (i = 1,\ldots,n) \quad (3.46)$$

for every point $(x_1,\ldots,x_n)$ on the path, divided by the total number of paths from $(0,\ldots,0)$ to $(k_1,\ldots,k_n)$.

To illustrate this point, we first consider the case $n = 2$. In this case probability (3.45) reduces to

$$\Pr\{X_{k2} - X_{k1} < K - k_1,\ k = 1,\ldots,k_1 + k_2 \mid N_1(C) = k_1, N_2(C) = k_2\}. \quad (3.47)$$

Now every sample path of $\{(X_{k1}, X_{k2})\}$ corresponds to a lattice path from $(0,0)$ to (k_1,k_2) (see Figure 3.1). More specifically, a horizontal step corresponds to an arrival in I_1, or a departure in I_2, and a vertical step to an arrival in I_2. Therefore, the number of customers in the system at the point (x_1,x_2) equals $k_1 - x_1 + x_2$, and this must be smaller than K for any x_1 and x_2, or $x_2 - x_1 < K - k_1$. Since every sample path of $\{(X_{k_1}, X_{k2})\}$ is

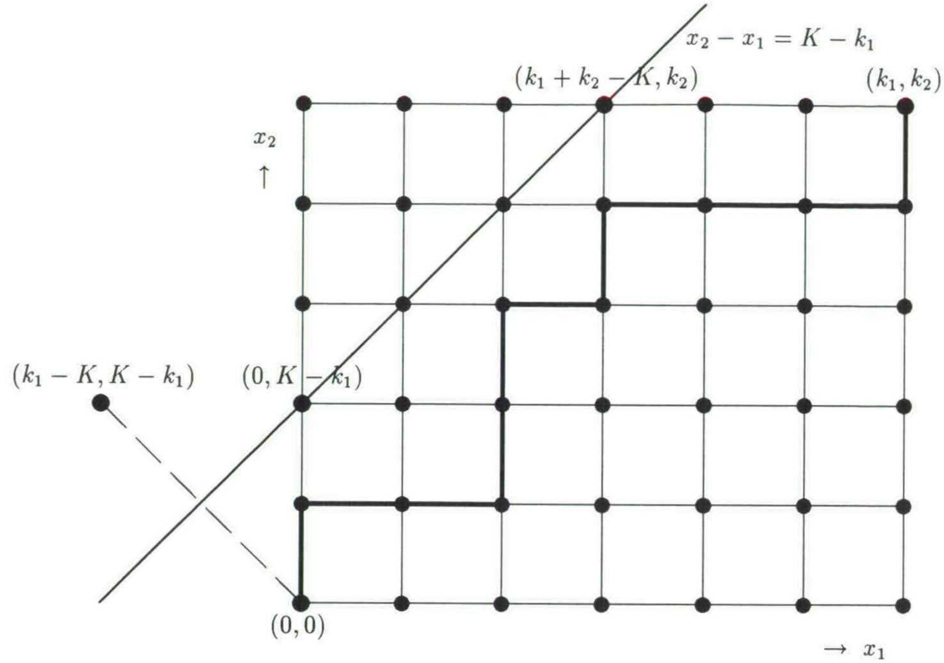

Figure 3.1: An example of a sample path of $\{(X_{k1}, X_{k2})\}$ ($K = 8$, $k_1 = 6$, $k_2 = 5$)

equally likely, probability (3.47) equals the number of lattice paths that remain below the line $x_2 - x_1 = K - k_1$, divided by the total number of paths $\binom{k_1+k_2}{k_1}$ (see Figure 3.1). We can count the number of paths remaining below this line by using the so-called principle of reflection (see e.g. [Feller 1968], p. 72, Lemma).

Proposition 3.1 *The number of paths from (a_1, a_2) to (b_1, b_2) which touch or cross the line l is equal to the number of paths from (a_1', a_2') to (b_1, b_2), with (a_1', a_2') the mirror image of (a, b) with respect to l.*

So according to Proposition 3.1, the number of paths from $(0,0)$ to (k_1, k_2) which touch or cross the line $x_2 - x_1 = K - k_1$ equals the number of paths from $(k_1 - K, K - k_1)$ to (k_1, k_2), or $\binom{k_1+k_2}{K}$ (see Figure 3.1). Hence the number of paths remaining below this line equals $\binom{k_1+k_2}{k_1} - \binom{k_1+k_2}{K}$ (for $k_1 + k_2 \geq K$; if $k_1 + k_2 < K$ then all paths automatically remain below this line), and it follows from (3.45) and (3.47) that

$$\Pr\{T_K^{(C)} > 2C \mid N_1(C) = k_1, N_2(C) = k_2\} = \begin{cases} 1 - \dfrac{\binom{k_1+k_2}{K}}{\binom{k_1+k_2}{k_1}} & \text{if } k_1 + k_2 \geq K; \\ 1 & \text{if } k_1 + k_2 < K. \end{cases} \tag{3.48}$$

Returning to the case of general n, we can use the following result from combinatorics, which can be seen as a generalisation of the principle of reflection; see e.g. [McMahon 1915] (p. 133), [Mohanty 1979] (p. 39, Theorem 3), or [Böhm et al. 1993] (Proposition 1).

Proposition 3.2 *The number of paths from* $\mathbf{a} := (a_1, \ldots, a_n)$ *to* $\mathbf{b} := (b_1, \ldots, b_n)$ *such that every point on the path satisfies* $x_1 \geq x_2 \geq \cdots \geq x_n$ *is given by*

$$\left(\sum_{i=1}^{n}(b_i - a_i)\right)!\ \det(C_n(\mathbf{a}, \mathbf{b})),$$

with

$$(C_n(\mathbf{a}, \mathbf{b}))_{ij} = \begin{cases} \dfrac{1}{(b_i - a_j - i + j)!} & \text{if } b_i - a_j \geq i - j \\ 0 & \text{if } b_i - a_j < i - j \end{cases} \qquad (i, j = 1, \ldots, n). \tag{3.49}$$

Using the transformation

$$\begin{aligned} y_1 &= x_1; \\ y_2 &= x_2 - (K - 1 - k_1) = x_2 + k_1 - K + 1; \\ y_3 &= x_3 - (K - 1 - k_1) - (K - 1 - k_2) = x_3 + k_1 + k_2 - 2(K - 1); \\ &\cdots \\ y_n &= x_n - (K - 1 - k_1) - \cdots - (K - 1 - k_{n-1}) = x_n + k_{1;n-1} - (n-1)(K-1), \end{aligned} \tag{3.50}$$

the conditions (3.44) reduce to $y_1 \geq y_2 \geq \cdots \geq y_n$. As a result, we can now apply Proposition 3.2 by setting

$$a_i := k_{1;i-1} - (i-1)(K-1), \quad b_i := k_{1;i} - (i-1)(K-1) \quad (i = 1, \ldots, n). \tag{3.51}$$

It follows that probability (3.45) is equal to

$$\frac{k_{1;n}!\ \det(C_n(\mathbf{k}))}{\binom{k_{1;n}}{k_1, \ldots, k_n}} = \left(\prod_{i=1}^{n} k_i!\right) \det(C_n(\mathbf{k})), \tag{3.52}$$

with

$$(C_n(\mathbf{k}))_{ij} = \begin{cases} \dfrac{1}{\big(k_{j;i} - (i-j)K\big)!} & (i = 1, \ldots, n;\ j = 1, \ldots, i); \\ \dfrac{1}{K!} & (i = 1, \ldots, n-1;\ j = i+1); \\ \dfrac{1}{\big((j-i)K - k_{i+1;j-1}\big)!} & (i = 1, \ldots, n-2;\ j = i+2, \ldots, n), \end{cases} \tag{3.53}$$

where $(C_n(\mathbf{k}))_{ij} = 0$ if the argument of the factorial is negative.

Combining (3.44), (3.45) and (3.52) leads to

Theorem 3.3 *The distribution of* $T_K^{(C)}$ *at integer multiples of* C *is given by*

$$\bar{F}_{T_K^{(C)}}(nC) = e^{-\lambda n C} \sum_{\substack{k_i = 0, \ldots, K-1; \\ i = 1, \ldots, n}} (\lambda C)^{k_{1;n}} \det(C_n(\mathbf{k})) \quad (n = 1, 2, \ldots),$$

with $C_n(\mathbf{k})$ *given by (3.53).*

For numerical purposes, this expression requires the computation of K^n determinants of order n. In Appendix 3.B we show that $\bar{F}_{T_2^{(C)}}(nC)$ reduces to the expression in Theorem 3.1(i) for $t = nC$, which turns out to be remarkably difficult.

3.3.3 Closed-form expressions

In this subsection we derive a closed-form expression for $\bar{F}_{T_K^{(C)}}(nC + t)$ as well as for $E\{T_K^{(C)}\}$. Define

$$\begin{aligned} J_i(t) &:= [(i-1)C, (i-1)C + t) \quad (i = 1, 2, \ldots;\ 0 < t \leq C); \\ K_i(t) &:= [(i-1)C + t, iC) \quad (i = 1, 2, \ldots;\ 0 < t \leq C); \\ \mathbf{l} &:= (l_1, \ldots, l_n); \\ \mathbf{m} &:= (m_1, \ldots, m_n); \\ E_n &:= \{\mathbf{N}(t) = \mathbf{l},\ \mathbf{N}(C) - \mathbf{N}(t) = \mathbf{m},\ N_{n+1}(t) = l_{n+1}\} \end{aligned}$$

(we omit the dependence of E_n on $\mathbf{l}$ and $\mathbf{m}$ for ease of notation). Note that $I_i = J_i(t) \cup K_i(t)$ $(i = 1, 2, \ldots;\ 0 < t \leq C)$, i.e., we chop every interval I_i into a left-hand part of length t and a right-hand part of length $C - t$. Now conditional on E_n, since customers that arrive in $J_i(t)$ (respectively $K_i(t)$) depart in $J_{i+1}(t)$ (respectively $K_{i+1}(t)$), we can decompose the process $\{X(s)\}$ on $[0, nC + t)$ into two independent parts: one on $\cup_{i=1}^{n+1} J_i(t)$ (the left-hand intervals) and one on $\cup_{i=1}^{n} K_i(t)$ (the right-hand intervals). The conditional independence of the two "subprocesses" follows by observing that $\{X(s),\ s \in K_i(t)\}$ only depends on $\{X(s),\ s \in J_i(t)\}$ via $N_i(t)$. Thus,

$$\begin{aligned} &\Pr\{X(s) < K,\ 0 \leq s \leq nC + t \mid E_n\} \\ = \ &\Pr\{X(s) < K,\ s \in \left(\cup_{i=1}^{n+1} J_i(t)\right) \bigcup \left(\cup_{i=1}^{n} K_i(t)\right) \mid E_n\} \\ = \ &\Pr\{X(s) < K,\ s \in \cup_{i=1}^{n+1} J_i(t) \mid E_n\} \cdot \Pr\{X(s) < K,\ s \in \cup_{i=1}^{n} K_i(t) \mid E_n\}. \quad (3.54) \end{aligned}$$

Both probabilities on the right-hand side of (3.54) can be computed in a similar fashion as probability (3.45). Consider the left-hand intervals $J_i(t)$ $(i = 1, \ldots, n+1)$. Given E_n, the number of customers at the start of $J_i(t)$ is equal to $l_{i-1} + m_{i-1}$, during $J_i(t)$ there are l_{i-1} departures and l_i arrivals, and hence the number of customers at the end of $J_i(t)$ is equal to $m_{i-1} + l_i$. Therefore, the left-hand probability in (3.54) can be written as

$$\begin{aligned} &\Pr\{X(s) < K,\ s \in \cup_{i=1}^{n+1} J_i(t) \mid E_n\} \\ = \ &\Pr\{l_{i-1} + m_{i-1} + X_{ki} - X_{k,i-1} < K,\ k = 1, \ldots, l_{1;n+1},\ i = 1, \ldots, n \mid E_n\} \\ = \ &\Pr\{X_{ki} - X_{k,i-1} < K - l_{i-1} - m_{i-1},\ k = 1, \ldots, l_{1;n+1},\ i = 1, \ldots, n \mid E_n\}, \quad (3.55) \end{aligned}$$

with $l_{1;n} := \sum_{i=1}^{n} l_i$. Since every sample path of $\{\mathbf{X}_k\}$ is equally likely (see Theorem 3.2), we see that probability (3.55) equals the number of paths from $(0, \ldots, 0)$ to $(l_1, \ldots, l_{n+1})$ that satisfy the conditions

$$x_i - x_{i-1} < K - (l_{i-1} + m_{i-1}) \quad (i = 1, \ldots, n+1) \qquad (3.56)$$

for every point $(x_1, \ldots, x_{n+1})$ on the path, divided by the total number of paths from $(0, \ldots, 0)$ to $(l_1, \ldots, l_{n+1})$. The transformation

$$y_i = x_i + \sum_{j=1}^{i-1} (l_j + m_j) - (i-1)(K-1) \quad (i = 1, \ldots, n+1),$$

enables us to apply Proposition 1, and it follows that (3.55) equals

$$\frac{l_{1;n+1}!\ \det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l))}{\binom{l_{1;n+1}}{l_1,\ldots,l_{n+1}}} = \left(\prod_{i=1}^{n+1} l_i!\right)\det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l)), \tag{3.57}$$

with (for $i = 1,\ldots,n+1$)

$$a_i^l := \sum_{j=1}^{i-1}(l_j+m_j)-(i-1)(K-1),\ b_i^l := \sum_{j=1}^{i}(m_{j-1}+l_j)-(i-1)(K-1). \tag{3.58}$$

Analogously, using the transformation

$$y_i = x_i + \sum_{j=1}^{i}(m_{j-1}+l_j)-(i-1)(K-1) \quad (i=1,\ldots,n), \tag{3.59}$$

it follows that the right-hand probability in (3.54) equals

$$\frac{m_{1;n}!\ \det(C_n(\mathbf{a}^r,\mathbf{b}^r))}{\binom{m_{1;n}}{m_1,\ldots,m_n}} = \left(\prod_{i=1}^{n} m_i!\right)\det(C_n(\mathbf{a}^r,\mathbf{b}^r)), \tag{3.60}$$

with $m_{1;n} := \sum_{i=1}^n m_i$ and (for $i = 1,\ldots,n$)

$$a_i^r := \sum_{j=1}^{i}(m_{j-1}+l_j)-(i-1)(K-1),\ b_i^r := \sum_{j=1}^{i}(l_j+m_j)-(i-1)(K-1). \tag{3.61}$$

We are now ready to present the complete distribution function.

Theorem 3.4 *The distribution of $T_K^{(C)}$ is given by (3.35), Theorem 3.3 and*

$$\bar{F}_{T_K^{(C)}}(nC+t) = e^{-\lambda(nC+t)} \sum_{\substack{l_i=0,\ldots,K-1-m_{i-1};\ i=1,\ldots,n+1 \\ m_i=0,\ldots,K-1-l_i;\ i=1,\ldots,n}} (\lambda t)^{l_{1;n+1}}\left((\lambda(C-t)\right)^{m_{1;n}}\det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l))\det(C_n(\mathbf{a}^r,\mathbf{b}^r)),$$

for $n = 1,2,\ldots$ and $0 < t < C$, with $C_n(\mathbf{a}^l,\mathbf{b}^l)$ and $C_n(\mathbf{a}^r,\mathbf{b}^r)$ given by (3.49), (3.58) and (3.61).

Proof. Conditioning on E_n and using (3.54), (3.57) and (3.60) leads to

$$\begin{aligned}\bar{F}_{T_K^{(C)}}(nC+t) &= \sum_{\substack{l_i=0,\ldots,K-1-m_{i-1};\ i=1,\ldots,n+1 \\ m_i=0,\ldots,K-1-l_i;\ i=1,\ldots,n}} \left(\prod_{i=1}^{n+1} e^{-\lambda t}\frac{(\lambda t)^{l_i}}{l_i!}\right)\cdot\left(\prod_{i=1}^{n} e^{-\lambda(C-t)}\frac{(\lambda(C-t)^{m_i}}{m_i!}\right)\cdot \\ &\quad \cdot\left(\prod_{i=1}^{n+1} l_i!\right)\det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l))\cdot\left(\prod_{i=1}^{n} m_i!\right)\det(C_n(\mathbf{a}^r,\mathbf{b}^r)),\end{aligned}$$

from which the desired result follows. □

To elaborate $E\{T_K^{(C)}\}$ we need the following lemma.

Lemma 3.2 *For $0 \le t \le C$ and $l, m = 0, 1, \ldots$*

(i) $\displaystyle\int_0^C e^{-\lambda t}(\lambda t)^l(\lambda(C-t))^m\,dt = \frac{1}{\lambda}\,\frac{(\lambda C)^{l+m+1}}{(l+m+1)!}\,l!\,m!\,M(-\lambda C, l+1, l+m+2)$

(ii) $\displaystyle = \frac{1}{\lambda}\left((\lambda C)^m\sum_{i=0}^{m}(l+i)!\binom{m}{i}\left(-\frac{1}{\lambda C}\right)^i - e^{-\lambda C}(\lambda C)^l\sum_{i=0}^{l}(m+i)!\binom{l}{i}\left(\frac{1}{\lambda C}\right)^i\right).$

Here $M(a,b,z)$ is a confluent hypergeometric function defined by

$$M(a,b,z) := \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \tag{3.62}$$

with

$$(a)_0 := 1,\ (a)_n := a(a+1)\cdots(a+n-1)\quad (n=1,2,\ldots) \tag{3.63}$$

("Kummer's function"; see [Abramowitz&Stegun 1965], 13.1.2).

Proof. Relation (i) follows from formula 13.2.1 in [Abramowitz&Stegun 1965], (ii) from induction and partial integration. □

Theorem 3.5 *The mean of $T_K^{(C)}$ is given by*

$$E\{T_K^{(C)}\} = \frac{1}{\lambda k!}\sum_{k=1}^{K}\gamma(k,\lambda C) + \sum_{n=1}^{\infty}e^{-n\lambda C}\sum_{\substack{l_i=0,\ldots,K-1-m_{i-1};\\ i=1,\ldots,n+1\\ m_i=0,\ldots,K-1-l_i;\ i=1,\ldots,n}} f(l_{1;n+1}, m_{1;n})\det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l))\det(C_n(\mathbf{a}^r,\mathbf{b}^r)),$$

with $C_n(\mathbf{a}^l,\mathbf{b}^l)$ and $C_n(\mathbf{a}^r,\mathbf{b}^r)$ given by (3.49), (3.58) and (3.61),

$$f(l,m) := \frac{(-1)^m}{\lambda}\left(\sum_{i=0}^{m}(l+m-i)!\binom{m}{i}(-\lambda C)^i - e^{-\lambda C}\sum_{i=0}^{l}(l+m-i)!\binom{l}{i}(\lambda C)^i\right), \tag{3.64}$$

and

$$\gamma(a,x) := \int_0^x e^{-t}t^{a-1}\,dt \quad (a, x > 0) \tag{3.65}$$

the incomplete gamma function (see [Abramowitz&Stegun 1965], 6.5.2).

Proof. Clearly,

$$E\{T_K^{(C)}\} = \int_0^{\infty}\bar{F}_{T_K^{(C)}}(t) = \sum_{n=0}^{\infty}\int_0^C \bar{F}_{T_K^{(C)}}(nC+t)\,dt. \tag{3.66}$$

For $n = 0$ we have by (3.35) that

$$\int_0^C F_{T_K^{(C)}}(t)\,dt = \int_0^C\sum_{k=0}^{K-1}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\,dt = \frac{1}{\lambda k!}\sum_{k=0}^{K-1}\int_0^{\lambda C}e^{-u}u^k\,du = \frac{1}{\lambda k!}\sum_{k=1}^{K}\gamma(k,\lambda C), \tag{3.67}$$

while for $n > 0$ the result follows from Theorem 3.4 and Lemma 3.2. □

This expression is obviously not suited for numerical purposes, as the number of terms in the third summation grows exponentially with n. However, in the next subsection we will show that $\bar{F}_{T_K^{(C)}}(t)$ can be closely approximated by an exponential function already for small values of n. This enables us to develop accurate and efficient approximations for $E\{T_K^{(C)}\}$.

λ	K	$c_1^{(1)}(K)$	$c_1^{(2)}(K)$	$c_1^{(3)}(K)$	$c_1^{(4)}(K)$
3	2	1.94395	1.96111	1.94999	1.94955
	3	1.23306	1.22782	1.21968	1.21967
	4	0.74921	0.73106	0.72713	0.72723
	5	0.42755	0.40900	0.40765	0.40771
	6	0.22487	0.21211	0.21181	0.21183
	7	0.10725	0.10059	0.10056	0.10057
	8	0.04588	0.04312	0.04312	0.04313
	9	0.01752	0.01658	0.01659	0.01659
	10	0.00599	0.00572	0.00572	0.00572
5	2	3.63476	3.69301	3.67645	3.67178
	3	2.63753	2.67542	2.65648	2.65367
	4	1.88869	1.89606	1.88082	1.87955
	5	1.32246	1.30735	1.29727	1.29684
	6	0.89725	0.87152	0.86589	0.86582
	7	0.58420	0.55781	0.55518	0.55521
	8	0.36135	0.34020	0.33921	0.33924
	9	0.21022	0.19612	0.19584	0.19585
	10	0.11401	0.10603	0.10597	0.10598

Table 3.1: $c_1^{(n)}(K)$ converges to a constant

3.3.4 Approximations and numerical results

The formulas in Theorems 3.3 and 3.4 are only useful to compute $\bar{F}_{T_K^{(C)}}(t)$ for small values of n, say $n \leq 3$ (for example, the computation time for $\bar{F}_{T_{10}^{(C)}}(5)$ is already in the order of hours on a Pentium PC). We now show that $\bar{F}_{T_K^{(C)}}(t)$ converges very rapidly to an exponential function, and use this to construct an accurate approximation for $E\{T_K^{(C)}\}$.

Define

$$c_1^{(n)}(K) := \ln \bar{F}_{T_K^{(C)}}(nC) - \ln \bar{F}_{T_K^{(C)}}((n+1)C) \quad (n = 0, 1, \ldots); \tag{3.68}$$

$$c_2^{(n)}(K) := e^{c_1^{(n)}} \bar{F}_{T_K^{(C)}}(nC) \quad (n = 0, 1, \ldots). \tag{3.69}$$

From Table 3.1 we see that $c_1^{(n)}(K)$ converges rapidly to a constant, implying that $\bar{F}_{T_K^{(C)}}(t)$ converges rapidly to an exponential function. Consequently, $\bar{F}_{T_K^{(C)}}(t)$ can be closely approximated by an exponential function for any value of K and already for $n \geq 2$. More formally, we have that

$$\lim_{t\to\infty} \left(\bar{F}_{T_K^{(C)}}(t) - c_2(K) e^{-c_1(K)t} \right) = 0, \tag{3.70}$$

with

$$c_1(K) := \lim_{n\to\infty} c_1^{(n)}(K), \quad c_2(K) := \lim_{n\to\infty} c_2^{(n)}(K). \tag{3.71}$$

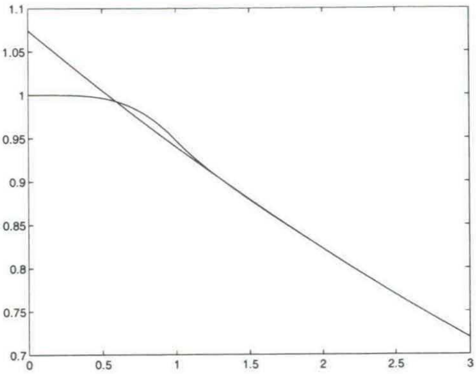

Figure 3.2: $\bar{F}(t)$ and $c_2^{(2)}e^{-c_1^{(2)}t}$ (λ=2,K=5)

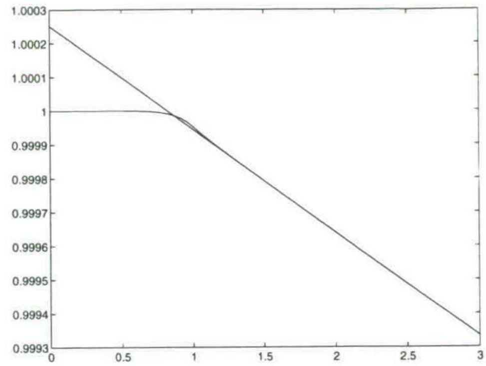

Figure 3.3: $\bar{F}(t)$ and $c_2^{(2)}e^{-c_1^{(2)}t}$ (λ=2,K=10)

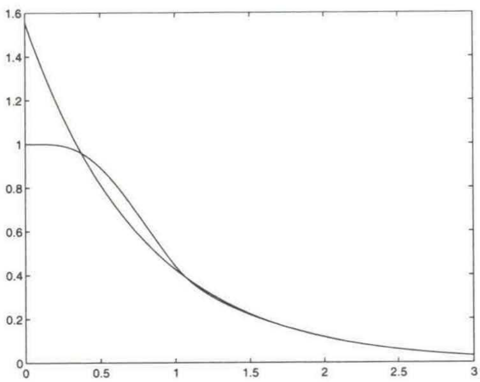

Figure 3.4: $\bar{F}(t)$ and $c_2^{(2)}e^{-c_1^{(2)}t}$ (λ=5,K=5)

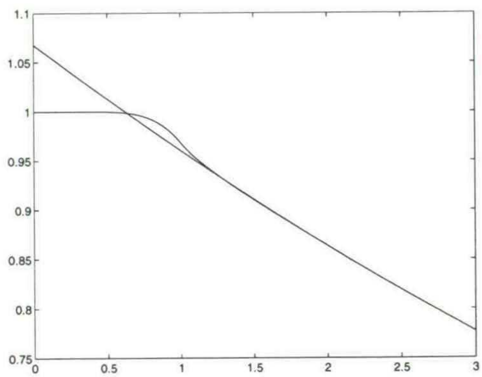

Figure 3.5: $\bar{F}(t)$ and $c_2^{(2)}e^{-c_1^{(2)}t}$ (λ=5,K=10)

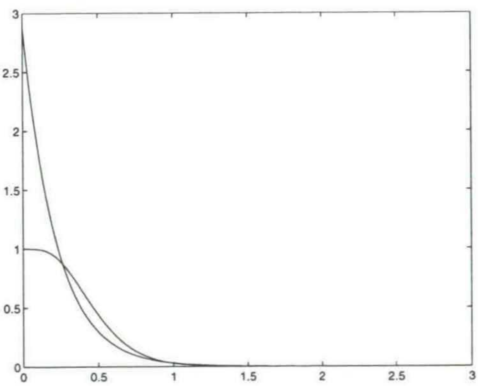

Figure 3.6: $\bar{F}(t)$ and $c_2^{(2)}e^{-c_1^{(2)}t}$ (λ=10,K=5)

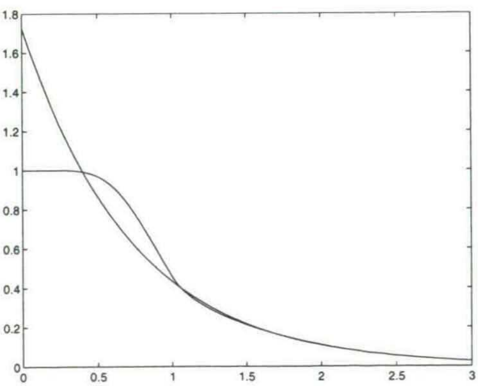

Figure 3.7: $\bar{F}(t)$ and $c_2^{(2)}e^{-c_1^{(2)}t}$ (λ=10,K=10)

λ	K	$E^{\text{sim}}\{T_K^{(C)}\}$	$\hat{E}\{T_K^{(C)}\}$	(sec.)	$\hat{g}_{\text{TD}}(K)$
2	1	0.5013±0.0031	0.5	(0)	4
	2	1.0778±0.0054	1.0782	(0)	2
	3	1.9783±0.0101	1.9700	(1)	1.4924
	4	3.7603±0.0205	3.7532	(1)	1.4671
	5	8.0471±0.0466	8.0229	(3)	1.6261
5	1	0.2005±0.0012	0.2	(0)	25
	2	0.4013±0.0018	0.4014	(0)	12.4746
	3	0.6123±0.0024	0.6111	(1)	8.2730
	4	0.8494±0.0032	0.8480	(1)	6.1792
	5	1.1514±0.0047	1.1500	(3)	5
	6	1.5890±0.0071	1.5849	(5)	4.3690
	7	2.2799±0.0111	2.2768	(7)	4.1216
	8	3.4786±0.0183	3.4688	(11)	4.1352
	9	5.6847±0.0316	5.6746	(16)	4.2951
	10	10.0379±0.0582	10.0549	(23)	4.5027
10	1	0.1003±0.0006	0.1	(0)	100
	2	0.1994±0.0009	0.2000	(0)	49.9991
	3	0.3005±0.0011	0.3001	(1)	33.3287
	4	0.4009±0.0013	0.4004	(2)	24.9853
	5	0.5022±0.0014	0.5018	(3)	19.9649
	6	0.6067±0.0017	0.6061	(5)	16.5995
	7	0.7194±0.0020	0.7174	(7)	14.1817
	8	0.8432±0.0025	0.8427	(11)	12.3733
	9	0.9939±0.0032	0.9933	(17)	11.0067
	10	1.1864±0.0042	1.1863	(24)	10
	11	1.4492±0.0057	1.4475	(32)	9.3091
	12	1.8222±0.0080	1.8169	(44)	8.8992
	13	2.3586±0.0112	2.3595	(58)	8.7285
	14	3.1928±0.0163	3.1857	(77)	8.7444
	15	4.4999±0.0241	4.4907	(99)	8.8866

Table 3.2: $E\{T_K^{(C)}\}$ and $g_{\text{TD}}(K)$ ($\lambda = 2, 5, 10$)

We were not able to derive analytical expressions for the parameters $c_1(K)$ and $c_2(K)$. In Figures 3.2–3.7 we plot $\bar{F}_{T_K^{(C)}}(t)$ and $c_2^{(2)}(K)e^{-c_1^{(2)}(K)t}$ for $t \leq 3$, for $\lambda = 2, 5, 10$ and $K = 5, 10$ (the index $T_K^{(C)}$ and the argument K are omitted for ease of notation). It is clear that the approximation is very accurate already for, say, $t \geq 2$. Note that in computing $\bar{F}_{T_K^{(C)}}(nC + t)$ from Theorem 3.4 for fixed n and a range of values for t $(0 < t < C)$ the determinants need to be evaluated only once, since they do not depend on t. Also $c_2(K) > 1$, whence the exponential function is not a distribution function.

Based on the foregoing we propose the following approximation for $\bar{F}_{T_K^{(C)}}(t)$:

$$\hat{\bar{F}}_{T_K^{(C)}}(t) = c_2^{(2)}(K)e^{-c_1^{(2)}(K)t} \quad (t \geq 2C), \tag{3.72}$$

with $c_1^{(2)}$ and $c_2^{(2)}$ given by (3.68) and (3.69), respectively. Using Theorem 3.5 and (3.72) we approximate $E\{T_K^{(C)}\}$ by

$$\begin{aligned} \hat{E}\{T_K^{(C)}\} &= \int_0^{2C} \bar{F}_{T_K^{(C)}}(t)\, dt + \int_{2C}^{\infty} \hat{\bar{F}}_{T_K^{(C)}}(t)\, dt \\ &= \frac{1}{\lambda k!}\sum_{k=1}^{K} \gamma(k, \lambda C) + \int_0^C \bar{F}_{T_K^{(C)}}(C + t)\, dt + \frac{c_2^{(2)}(K)}{c_1^{(2)}(K)} e^{-2C\, c_1^{(2)}(K)}, \end{aligned} \tag{3.73}$$

where the integral can be written as a sum as in Theorem 3.5.

In Table 3.2 we compare $\hat{E}\{T_K^{(C)}\}$ with the simulation value $E^{\text{sim}}\{T_K^{(C)}\}$ (and its 95% confidence interval) after 10^5 runs, for $\lambda = 2, 5, 10$. In brackets we give the computation time in seconds on a 486 PC for $\hat{E}\{T_K^{(C)}\}$. The approximation performs very well, and always falls within the 95% confidence interval. As expected, the computation time increases exponentially with K. In the last column of Table 3.2 we evaluate the expected costs of the TD-policy from (3.23) for cost parameters $a_B = \lambda$, $b_B = 0$ and $b_I = 1$. We see that the optimal control-limit K^* equals 4 for $\lambda = 2$, 7 for $\lambda = 5$ and 13 for $\lambda = 10$. More extensive numerical results for the TD-policy as well as for the GCG-policy will be given in section 3.6.

3.4 The Improved Generalized Critical-Group policy

Although the GCG-policy is a very flexible policy, it has an obvious shortcoming: whenever a batch service is started there is no delay-limit that expires at that time. Consequently, it is better to wait for the next delay-limit to expire and include (possible) additional customers in the batch. The point is that a batch service should not be triggered by a customer arrival, but by a customer "departure" (i.e., the expiration of a delay-limit). This brings us to the following policy: start a batch service at the first epoch that a delay-limit expires after the moment at which the size of the critical group has reached the level K (recall that the critical group consists of the waiting customers with a residual delay-limit of at most C time units). This is equivalent to waiting until the size of the critical group reaches the level K and starting a batch service when the delay-limit of the "oldest" customer in the critical group expires. Since this policy can only improve upon the GCG-policy, we refer to it as the Improved Generalized Critical-Group (IGCG) policy and to the case $C = D$ as the Improved Total-Demand (ITD) policy.

Clearly, the epochs at which a delay-limit expires are $\{B_1+D, B_2+D, \ldots\}$. The critical group at epoch $B_n + D$ consists of the arrivals during $[B_n, B_n + C]$, so that the cycle length for the IGCG-policy is given by

$$\begin{aligned} S_{\text{IGCG}} &= \min_{n=1,2,\ldots} \big\{B_n + D : N(B_n, B_n + C) + 1 \geq K\big\} \\ &= D + \min_{n=1,2,\ldots} \big\{B_n : N(B_n + C) \geq K + n - 1\big\}. \end{aligned} \tag{3.74}$$

It follows that $\Pr\{S_{\text{IGCG}} > t\} = 1$ for $t \leq D$ and

$$\begin{aligned} \Pr\{S_{\text{IGCG}} > t\} &= \Pr\{N(B_i + C) < K + i - 1;\ i = 1, \ldots, N(t - D)\} \\ &= \Pr\{B_K > B_1 + C, \ldots, B_{N(t-D)+K-1} > B_{N(t-D)} + C\} \end{aligned} \tag{3.75}$$

for $t > D$. Although (3.75) is similar to (3.34), S_{IGCG} cannot be expressed in terms of $T_K^{(C)}$. However, we can express S_{IGCG} in terms of $N_K^{(C)}$: since the $(N_K^{(C)})^{\text{th}}$ customer increases the size of the critical group to K, after which the batch service is started as soon as the delay-limit of the oldest customer in the critical group expires, we have that

$$S_{\text{IGCG}} = B_{N_K^{(C)}-K+1} + D = D + \sum_{i=1}^{N_K^{(C)}-K+1} A_i. \tag{3.76}$$

But now we cannot apply Wald's equation like in (3.18); contrary to $N_K^{(C)}$, $N_K^{(C)} - K + 1$ is not a stopping time for $\{A_i\}$ because $\{N_K^{(C)} = n + K - 1\}$ depends on $\{A_{n+1}, \ldots, A_{n+K-1}\}$.

The additional difficulties associated with the IGCG-policy are caused by the fact that S_{IGCG} is not a stopping time for $\{N(t)\}$, i.e., it does not hold that the event $\{S_{\text{IGCG}} \leq t\}$ only depends on $\{N(u), 0 \leq u \leq t\}$, which is easily verified from (3.74). In fact, all other policies π considered in this chapter do have the property that S_π is a stopping time for $\{N(t)\}$. To overcome the difficulties, we construct an approximate embedded Markov chain on departure epochs in the next subsection.

3.4.1 An embedded Markov chain approximation

Consider a M/D/∞ queue with service times C and define for $n = 1, 2, \ldots$

$$\begin{aligned} X_n^{(C)} &:= \text{number of customers that the } n^{\text{th}} \text{ departing customer leaves behind;} \\ A_n^{(C)} &:= \text{time between } (n-1)^{\text{th}} \text{ and } n^{\text{th}} \text{ departure.} \end{aligned}$$

It is easily seen that

$$X_n^{(C)} = \begin{cases} N(B_n, B_n + C) & \text{if } X_{n-1}^{(C)} = 0 \\ X_{n-1}^{(C)} - 1 + N(B_{n-1} + C, B_n + C) & \text{if } X_{n-1}^{(C)} > 0 \end{cases} \quad (n = 1, 2, \ldots); \tag{3.77}$$

$$A_n^{(C)} = \begin{cases} A_n + C & \text{if } X_{n-1}^{(C)} = 0 \\ A_n & \text{if } X_{n-1}^{(C)} > 0 \end{cases} \quad (n = 1, 2, \ldots). \tag{3.78}$$

However, since $N(B_{n-1}+C, B_n+C)$ depends on A_n which in turn may depend on $\{X_{n-2}^{(C)}, X_{n-3}^{(C)}, \ldots\}$, the stochastic process $\{X_n^{(C)}\}$ is not a Markov chain. We approximate $\{X_n^{(C)}\}$ by treating it as an embedded Markov chain, i.e., we only use information on $X_{n-1}^{(C)}$ in determining the distribution of $X_n^{(C)}$ and $A_n^{(C)}$. Denote the approximate embedded Markov chain by $\{\hat{X}_n^{(C)}\}$ and the approximate transition times by $\{\hat{A}_n^{(C)}\}$, and suppose that $\hat{X}_{n-1}^{(C)} = i$ at time t. Then by (3.36) the arrival times of the present customers are uniformly distributed over $(t-C, t]$, so that the departure times are uniformly distributed over $(t, t+C]$. Hence the time until the next departure is the minimum order statistic of i uniformly distributed random variables over $[0, C]$, implying that

$$\begin{aligned}\Pr\{\hat{A}_n^{(C)} > x \mid \hat{X}_{n-1}^{(C)} = i\} &= \Pr\{A_1 > x \mid N(C) = i\} \\ &= \left(\frac{C-x}{C}\right)^i \quad (i > 0;\ 0 \le x \le C),\end{aligned} \tag{3.79}$$

and

$$E\{\hat{A}_n^{(C)} \mid \hat{X}_{n-1}^{(C)} = i\} = \frac{C}{i+1} \quad (i > 0). \tag{3.80}$$

The stochastic process $\{\hat{X}_n^{(C)}\}$ can be interpreted as follows. All arriving customers take place in a waiting room for exactly D time units, but their arrival time is discarded. At any departure epoch, the number of customers in the waiting room is checked and – using only the information that customers arrive according to a Poisson process and that the waiting customers are precisely the customers that have arrived during the last D time units – the distribution of the number of customers in the waiting room at the next departure epoch is determined.

Define the transition probabilities $p_{ij} := \Pr\{\hat{X}_n^{(C)} = j \mid \hat{X}_{n-1}^{(C)} = i\}$, then conditioning on $\hat{A}_n^{(C)}$ and using (3.77) and (3.79) yields

$$p_{ij} = \begin{cases} \displaystyle\int_{x=0}^{C} \frac{i}{C}\left(\frac{C-x}{C}\right)^{i-1} e^{-\lambda x}\frac{(\lambda x)^{j-i+1}}{(j-i+1)!}\,dx & \text{if } i > 0 \text{ and } j \ge i-1; \\ \displaystyle e^{-\lambda C}\frac{(\lambda C)^j}{j!} & \text{if } i = 0 \text{ and } j \ge 0. \end{cases} \tag{3.81}$$

Using partial integration it can be shown that

$$\begin{aligned} p_{10} &= \frac{1-e^{-\lambda C}}{\lambda C}; \\ p_{1j} &= p_{1,j-1} - e^{-\lambda C}\frac{(\lambda C)^{j-1}}{j!} \quad (j = 1, 2, \ldots); \\ p_{i,i-1} &= \frac{i}{\lambda C}(1 - p_{i-1,i-2}) \quad (i = 2, 3, \ldots); \\ p_{ij} &= p_{i,j-1} - \frac{i}{\lambda C} p_{i-1,j-1} \quad (i = 2, 3, \ldots;\ j = i, i+1, \ldots). \end{aligned} \tag{3.82}$$

This two-dimensional first-order difference equation has no simple general solution, except for the boundaries at $i = 1$ and $j = i-1$; there we obtain the explicit solutions

$$p_{1j} = \frac{1}{\lambda C}\left(1 - \sum_{n=0}^{j} e^{-\lambda C}\frac{(\lambda C)^n}{n!}\right) \quad (j = 0, 1, \ldots); \tag{3.83}$$

$$p_{i,i-1} = \frac{i!}{(-\lambda C)^i}\left(e^{-\lambda C} - \sum_{n=0}^{i-1}\frac{(-\lambda C)^n}{n!}\right) \quad (i = 1, 2, \ldots). \tag{3.84}$$

Although (3.82) can be used to calculate p_{ij} recursively, this turns out to give numerical problems already for moderate values of i (mainly because of the subtractions). However, we can eliminate the numerical instability by putting $p_{ij} = 0$ for $j > N$ with N sufficiently large and then calculate p_{ij} backwards. Rewriting (3.82) we arrive at the following computational scheme:

$$\begin{aligned} p_{i,N+1} &:= 0 \quad (i = 1, 2, \ldots, N+1); \\ p_{1j} &= p_{1,j+1} + e^{-\lambda C}\frac{(\lambda C)^j}{(j+1)!} \quad (j = 0, 1, \ldots, N); \\ p_{i,i+k} &= p_{i,i+(k+1)} + \frac{i}{\lambda C}p_{i-1,i-1+(k+1)} \quad (k = N\text{–}2, N\text{–}1, \ldots, -1;\ i = 2, 3, \ldots, N\text{–}k). \end{aligned} \tag{3.85}$$

Thus, we start with $k = j - i = N - 1$ and decrease k in every step. This scheme is numerically stable and converges very fast with decreasing k; as a rule of thumb, it suffices to set $N = 2n + 10$ in computing p_{ij} for $i, j = 0, 1, \ldots, n$.

In order to compute the average costs for the IGCG-policy, we define for $i = 0, 1, \ldots$ and $j = i+1, i+2, \ldots$

$t_{ij}^{(C)}$:= expected first passage time from state i to state j;

$u_{ij}^{(C)}$:= expected number of departures during first passage time from state i to state j;

$v_{ij}^{(C)}$:= expected overshoot of level j at first passage from state i to state j.

Now observe that under the IGCG-policy a batch service is started exactly $D - C$ time units after the first epoch at which a customer with a delay of C time units sees $K - 1$ or more "younger" customers behind him, and that this epoch corresponds to the first entrance time into $\{K-1, K, \ldots\}$ of $\{X_n^{(C)}\}$. Therefore,

$$\hat{E}\{S_{\text{IGCG}}\} = t_{0,K-1}^{(C)} + D - C; \tag{3.86}$$

$$\hat{E}\{Y_{\text{IGCG}}\} = \sum_{j=0}^{K-2} p_{0j}u_{j,K-1}^{(C)} = u_{0,K-1}^{(C)} - 1; \tag{3.87}$$

$$\hat{E}\{Z_{\text{IGCG}}\} = K + v_{0,K-1}^{(C)} + \lambda(D - C), \tag{3.88}$$

and using (3.4) we find that

$$\hat{g}_{\text{IGCG}}(K) = \frac{a_B + b_B\left(K + v_{0,K-1}^{(C)} + \lambda(D - C)\right) + b_I\left(u_{0,K-1}^{(C)} - 1\right)}{t_{0,K-1}^{(C)} + D - C}. \tag{3.89}$$

For $t^{(C)}_{i,K-1}$, $u^{(C)}_{i,K-1}$ and $v^{(C)}_{i,K-1}$ $(i = 0, 1, \ldots, K-2)$ we obtain a linear system of equations by conditioning on the next state. It follows from (3.78) and (3.80) that

$$\begin{aligned} t^{(C)}_{0,K-1} &= \frac{1}{\lambda} + C + \sum_{j=0}^{K-2} p_{0j} t^{(C)}_{j,K-1}; \\ t^{(C)}_{i,K-1} &= \frac{C}{i+1} + \sum_{j=i-1}^{K-2} p_{ij} t^{(C)}_{j,K-1} \quad (i = 1, \ldots, K-2). \end{aligned} \tag{3.90}$$

Furthermore, since departure epochs are transition epochs,

$$\begin{aligned} u^{(C)}_{0,K-1} &= 1 + \sum_{j=0}^{K-2} p_{0j} u^{(C)}_{j,K-1}; \\ u^{(C)}_{i,K-1} &= 1 + \sum_{j=i-1}^{K-2} p_{ij} u^{(C)}_{j,K-1} \quad (i = 1, \ldots, K-2) \end{aligned} \tag{3.91}$$

and, since the overshoot is $j - K + 1$ if $j > K - 1$ and 0 otherwise,

$$\begin{aligned} v^{(C)}_{0,K-1} &= \sum_{j=0}^{K-2} p_{0j} v^{(C)}_{j,K-1} + \sum_{j=K-1}^{\infty} (j - K + 1) p_{0j}; \\ v^{(C)}_{i,K-1} &= \sum_{j=i-1}^{K-2} p_{ij} v^{(C)}_{j,K-1} + \sum_{j=K-1}^{\infty} (j - K + 1) p_{ij} \quad (i = 1, \ldots, K-2). \end{aligned} \tag{3.92}$$

Solving (3.90)–(3.92), all linear systems of size $K-1$, gives the desired values for $t^{(C)}_{0,K-1}$, $u^{(C)}_{0,K-1}$ and $v^{(C)}_{0,K-1}$ needed in (3.89). Using the relations

$$\sum_{j=K-1}^{\infty} (j-K+1) p_{0j} = \lambda C - K + 1 - \sum_{j=0}^{K-2} (j-K+1) p_{0j}; \tag{3.93}$$

$$\sum_{j=K-1}^{\infty} (j-K+1) p_{ij} = \frac{\lambda C}{i+1} - K + i - \sum_{j=i-1}^{K-2} (j-K+1) p_{ij} \; (i = 1, \ldots, K-2) \tag{3.94}$$

to simplify (3.92), this only requires the computation of p_{ij} for $i, j = 0, 1, \ldots, K-2$ from (3.85) with $N = 2K+6$ (according to our rule of thumb).

3.4.2 Numerical results

To provide some insight into the quality of the embedded Markov chain approximation, we compare the approximate values for $E\{S_{\text{ITD}}\}$, $E\{Y_{\text{ITD}}\}$ and g_{ITD} with simulation values (and 95% confidence interval) after 10^5 runs (see Table 3.3). The accuracy of the approximation decreases with λ and K, and $\hat{g}_{\text{ITD}}$ overestimates the true value. This is due to the fact that the embedded Markov chain has a larger degree of randomness than the M/D/∞ process. However, since the shape of the cost function and the optimal value of K remain the same, the approximation is very useful in determining a good service policy. Also, the

λ	K	$E^{\text{sim}}\{S_{\text{ITD}}\}$	$\hat{E}\{S_{\text{ITD}}\}$	$E^{\text{sim}}\{Y_{\text{ITD}}\}$	$\hat{E}\{Y_{\text{ITD}}\}$	$g^{\text{sim}}_{\text{ITD}}(K)$	$\hat{g}_{\text{ITD}}(K)$
2	1	1.5007±0.0031	1.5	0	0	1.3327	1.3333
	2	1.7365±0.0052	1.7348	0.1574±0.0027	0.1565	1.2424	1.2431
	3	2.4100±0.0099	2.4241	0.9506±0.0096	0.9837	1.2243	1.2309
	4	4.0591±0.0204	4.1703	3.5114±0.0275	3.7409	1.3578	1.3766
	5	8.2797±0.0465	8.6239	11.1005±0.0760	11.7997	1.5822	1.6002
5	1	1.1999±0.0012	1.2	0	0	4.1671	4.1667
	2	1.2085±0.0014	1.2081	0.0073±0.0005	0.0068	4.1434	4.1442
	3	1.2399±0.0018	1.2394	0.0557±0.0018	0.0576	4.0774	4.0805
	4	1.3252±0.0027	1.3255	0.2427±0.0047	0.2599	3.9561	3.9683
	5	1.5053±0.0043	1.5184	0.7545±0.0102	0.8402	3.8229	3.8463
	6	1.8451±0.0067	1.9014	1.9180±0.0199	2.2096	3.7495	3.7918
	7	2.4787±0.0109	2.6161	4.3935±0.0378	5.0975	3.7896	3.8598
	8	3.6349±0.0182	3.9297	9.3793±0.0710	10.8740	3.9559	4.0395
	9	5.8135±0.0315	6.3954	19.4075±0.1326	22.3391	4.1984	4.2748
	10	10.1523±0.0582	11.2481	40.1935±0.1875	45.6936	4.4515	4.5069
10	1	1.1005±0.0006	1.1	0	0	9.0867	9.0909
	2	1.1006±0.0006	1.1000	0.0001±0.0001	0.0000	9.0862	9.0905
	3	1.1009±0.0006	1.1003	0.0007±0.0002	0.0006	9.0838	9.0886
	4	1.1024±0.0007	1.1016	0.0044±0.0005	0.0039	9.0754	9.0814
	5	1.1067±0.0008	1.1056	0.0186±0.0012	0.0180	9.0529	9.0614
	6	1.1174±0.0009	1.1162	0.0618±0.0026	0.0644	9.0046	9.0166
	7	1.1418±0.0013	1.1405	0.1753±0.0049	0.1897	8.9118	8.9341
	8	1.1868±0.0019	1.1899	0.4242±0.0087	0.4807	8.7835	8.8083
	9	1.2674±0.0027	1.2807	0.9217±0.0146	1.0812	8.6173	8.6522
	10	1.4034±0.0038	1.4365	1.8493±0.0236	2.2124	8.4435	8.5015
	11	1.6218±0.0055	1.6903	3.4829±0.0370	4.2046	8.3134	8.4037
	12	1.9592±0.0078	2.0908	6.1996±0.0571	7.5552	8.2684	8.3964
	13	2.4744±0.0111	2.7140	10.6123±0.0869	13.0401	8.3302	8.4894
	14	3.2862±0.0162	3.6848	17.9096±0.1321	21.9292	8.4931	8.6651
	15	4.5826±0.0241	5.2201	29.9962±0.1840	36.4093	8.7278	8.8906

Table 3.3: $E\{S_{\text{ITD}}\}$, $E\{Y_{\text{ITD}}\}$ and $g_{\text{ITD}}(K)$ ($\lambda = 2, 5, 10$)

approximation is very efficient in terms of computation time, e.g., it is much faster than the approximation for g_{ITD} (see section 3.4). Because the computation time increases only linearly with K, large values of λ and K are also feasible (but at the expense of a lower accuracy). As expected, the ITD-policy (EGCG-policy) is considerably better than the TD-policy (GCG-policy), especially for low values of K; the difference in costs tends to zero as K tends to infinity.

The reason that the accuracy of the approximation decreases rapidly with K is that the delay-limit is not treated as a constant but as a stochastic variable. Basically, the information on the residual delay-limits of the waiting customers at departure epochs is discarded and instead they are assumed to be uniformly distributed over $[0, D]$. The larger the value of K, the more information is discarded. It is not clear what the effective delay-limit distribution of an arbitrary customer is, i.e., the distribution of the time between arrival and departure of an arbitrary customer under the embedded Markov chain.

The determination of the optimal control parameters C^* and K^* is not a trivial matter, especially since C is not restricted to integer values. In Figures 3.8–3.11 we plot $\hat{g}_{\text{IGCG}}(C, K)$ as a function of C only by setting $K = K^*(C)$, the optimal value for K given C, for $\lambda \in \{2, 5, 10, 20\}$. The other parameters are $D = 1$, $a_B = \lambda$, $b_B = 0$ and $b_I = 1$. This results in cloud-like curves, where the non-differentiable points correspond to jumps in $K^*(C)$. We see that the optimal value of C is usually located between $\frac{1}{2}$ and 1; $C = 1$ corresponds to the ITD-policy. Moreover, all numerical experiments indicate that $\hat{g}_{\text{IGCG}}(C, K)$ is convex in C for fixed K, and convex in K for fixed C. Based on this we propose the following search procedure to find (C^*, K^*): use a bisection algorithm on K, and for fixed K use a bisection algorithm on C ("nested bisection"). It is easily verified that reversing the order of K and C (bisection on C and for fixed C bisection on K) does not always lead to an optimum.

3.5 A phase-type model

Most of the difficulties encountered so far are caused by the assumption of a constant delay-limit, which makes a Markov chain analysis impossible. In this section we avoid these difficulties by assuming a "phase-type" distribution for the delay-limit, so that the process can be modelled as a (multi-dimensional) continuous-time Markov chain. First we will give a complete analysis of the model with an exponentially distributed delay-limit (subsection 3.5.2). Next we will consider a more general model where the delay-limit has an Erlang(n, μ) distribution with mean $\frac{n}{\mu}$. Since the Erlang$(n, \frac{D}{n})$ distribution converges to its mean D as n tends to infinity, the "Erlang-n model" can be used as an approximation for the model with a constant delay-limit. Increasing n increases the accuracy of the approximation, at the expense of model complexity and computational effort (see also section 2.8). But apart from serving as an approximation, the Erlang-n model is interesting in its own right, and also useful in determining the sensitivity of the service model on the delay-limit distribution. We close this section with some computational results for the exponential model (subsection 3.5.3).

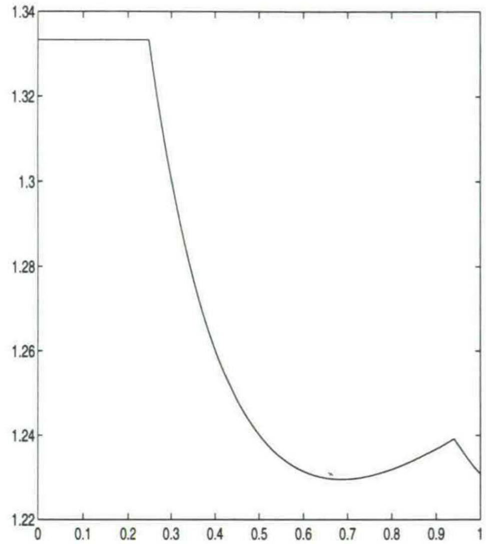

Figure 3.8: $\hat{g}_{\mathrm{IGCG}}(C, K^*(C))$ for $\lambda = 2$

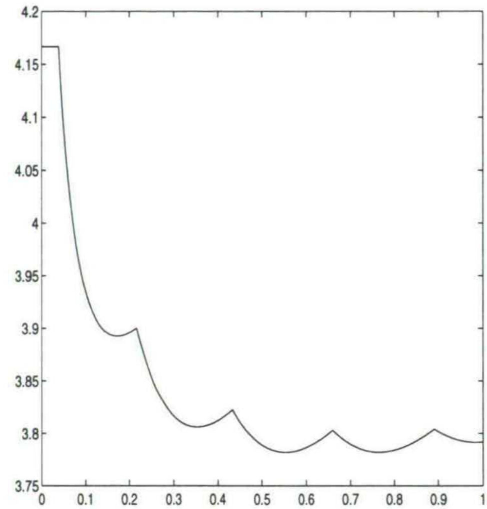

Figure 3.9: $\hat{g}_{\mathrm{IGCG}}(C, K^*(C))$ for $\lambda = 5$

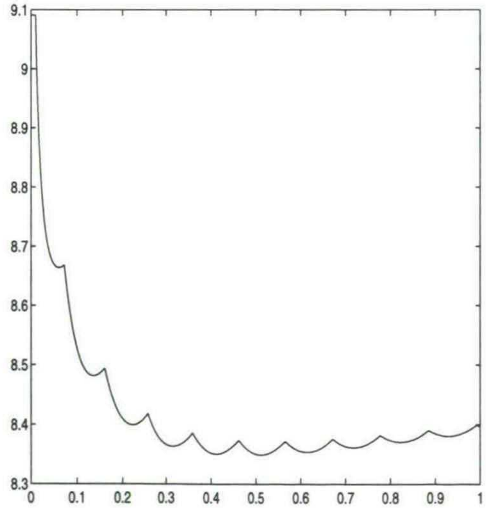

Figure 3.10: $\hat{g}_{\mathrm{IGCG}}(C, K^*(C))$ for $\lambda = 10$

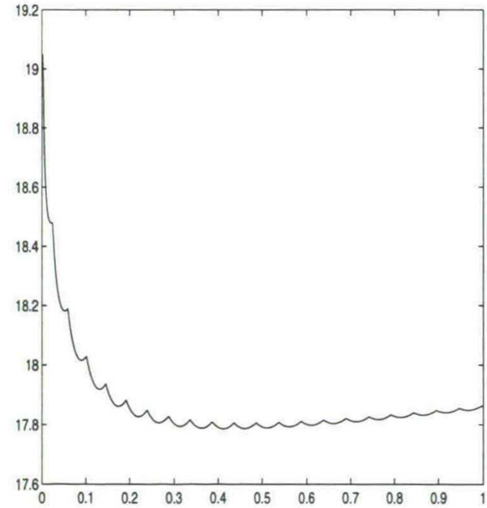

Figure 3.11: $\hat{g}_{\mathrm{IGCG}}(C, K^*(C))$ for $\lambda = 20$

3.5.1 An exponentially distributed delay-limit

The simplest example of a phase-type model is the case of an exponentially distributed delay-limit with mean $\frac{1}{\mu}$. In this case the total number of waiting customers provides a complete state description, because the residual delay-limit of any waiting customer is exponentially distributed with mean $\frac{1}{\mu}$. We will first compute the average costs for the TD-policy, and next the average costs for the ITD-policy that postpones the batch service until a delay-limit expires.

The TD-policy

Analogously to (3.23), the expected average costs for the TD-policy are given by

$$g_{\mathrm{TD}}^{\mathrm{exp}}(K) = \frac{a_B + b_B K + b_I\big(\lambda E\{T_K^{\mathrm{exp}}\} - K\big)}{E\{T_K^{\mathrm{exp}}\}} = \lambda b_I - \frac{(b_I - b_B)K - a_B}{E\{T_K^{\mathrm{exp}}\}}, \tag{3.95}$$

with T_K^{exp} the first entrance time into state K of a M/M/∞ queue with arrival rate λ and service rate μ. Now consider a general birth-death process with birth rates λ_i $(i = 0, 1, \ldots)$ and death rates μ_i $(i = 1, 2, \ldots)$, and define

$$T_{i,j} \ := \ \text{first passage time from state } i \text{ to state } j \quad (i, j = 1, 2, \ldots),$$

It can be shown that

$$T_K := T_{0,K} = \sum_{i=0}^{K-1} T_{i,i+1} \tag{3.96}$$

and

$$E\{T_{i,i+1}\} = \frac{1}{\lambda_i b_i} \sum_{j=0}^{i} b_j, \tag{3.97}$$

with

$$b_i := \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}$$

(see e.g. [Heyman&Sobel 1982], Theorem 4-12). Since the number of customers in a M/M/∞ queue is a birth-death process with

$$\lambda_{i-1} = \lambda, \quad \mu_i = i\mu, \quad \rho := \frac{\lambda}{\mu}, \quad b_i = \frac{\rho^i}{i!} \quad (i = 1, 2, \ldots), \tag{3.98}$$

it follows from (3.96) and (3.97) that

$$E\{T_K^{\mathrm{exp}}\} = \sum_{i=0}^{K-1} \left(\lambda \frac{\rho^i}{i!}\right)^{-1} \sum_{j=0}^{i} \frac{\rho^j}{j!} = \frac{1}{\lambda} \sum_{i=0}^{K-1} \frac{\sum_{j=0}^{i} \frac{\rho^j}{j!}}{\frac{\rho^i}{i!}}. \tag{3.99}$$

In the i^{th} term of the sum in (3.99) we recognize the inverse of $\pi_i^{\mathrm{M/M}/i/i}$, the probability of i customers in a M/M/i/i Erlang loss system. To explain this, consider a M/M/i/i Erlang loss system and define a regenerative cycle as the time between two consecutive epochs that a customer is lost. The average number of customers lost per unit time is given by

$\lambda\pi_i^{\mathrm{M/M}/i/i}$ and the cycle length corresponds to $T_{i,i+1}^{\mathrm{M/M}/\infty}$, the first passage time from i to $i+1$ in a M/M/∞ queue. Applying the Renewal Reward Theorem we find that

$$\lambda\pi_i^{\mathrm{M/M}/i/i} = \frac{1}{E\{T_{i,i+1}^{\mathrm{M/M}/\infty}\}} \quad (i = 0, 1, \ldots), \tag{3.100}$$

or

$$E\{T_{i,i+1}^{\mathrm{M/M}/\infty}\} = \frac{1}{\lambda\pi_i^{\mathrm{M/M}/i/i}} \quad (i = 0, 1, \ldots), \tag{3.101}$$

providing an elegant proof of (3.99).

The ITD-policy

The Improved Total-Demand (ITD) policy improves upon the TD-policy by postponing the batch service until a delay-limit expires, after the number of waiting customers has reached the level K (see also section 3.4). In this way the customers that arrive in the meantime can also be included in the batch service, while no additional individual services are needed. If we define, for a M/M/∞ queue,

f_i := expected time until the next departure in state i $(i = 0, 1, \ldots)$;
g_i := expected number of arrivals before the next departure in state i $(i = 0, 1, \ldots)$,

then it is easily seen that the expected average costs for the ITD-policy are given by

$$g_{\mathrm{ITD}}^{\mathrm{exp}}(K) = \frac{a_B + b_B(K + g_K) + b_I\left(\lambda E\{T_K^{\mathrm{exp}}\} - K\right)}{E\{T_K^{\mathrm{exp}}\} + f_K}. \tag{3.102}$$

Conditioning on the next event in state i, which is an arrival with probability $\frac{\lambda}{\lambda+i\mu}$ and a departure with probability $\frac{i\mu}{\lambda+i\mu}$, we obtain the following first-order difference equations:

$$f_i = \frac{1}{\lambda + i\mu} + \frac{\lambda}{\lambda + i\mu} f_{i+1} \quad (i = 0, 1, \ldots); \tag{3.103}$$

$$g_i = \frac{\lambda}{\lambda + i\mu}(1 + g_{i+1}) \quad (i = 0, 1, \ldots). \tag{3.104}$$

It immediately follows that

$$g_{i+1} = \left(1 + \frac{i}{\rho}\right) g_i - 1 \quad (i = 0, 1, \ldots); \tag{3.105}$$

$$f_i = \frac{1}{\lambda} g_i \quad (i = 0, 1, \ldots), \tag{3.106}$$

and hence it suffices to solve (3.105). Iterating (3.105) yields

$$g_{i+1} = \left(1 + \frac{1}{\rho}\right) \cdots \left(1 + \frac{i}{\rho}\right) g_0 - \sum_{n=1}^{i} \left(1 + \frac{n}{\rho}\right) \cdots \left(1 + \frac{i}{\rho}\right) - 1, \tag{3.107}$$

which, together with $\lim_{i\to\infty} g_i = 0$, leads to

$$\begin{aligned}
g_0 &= \lim_{i\to\infty} \frac{1 + \sum\limits_{n=1}^{i} \left(1+\frac{n}{\rho}\right)\cdots\left(1+\frac{i}{\rho}\right)}{\left(1+\frac{1}{\rho}\right)\cdots\left(1+\frac{i}{\rho}\right)} \\
&= 1 + \lim_{i\to\infty} \sum_{n=1}^{i} \left(\frac{\rho}{\rho+1}\cdots\frac{\rho}{\rho+n}\right) \\
&= 1 + \sum_{n=1}^{\infty} \frac{\rho^n}{(\rho+1)\cdots(\rho+n)}. \qquad (3.108)
\end{aligned}$$

If $\rho = 1$ then (3.108) reduces to $e-1$, which has an interesting interpretation: the expected number of arrivals before the first departure in a M/M/∞ queue with $\lambda = \mu$ is equal to $e-1$. Moreover, we have that

$$g_0 = 1 + \sum_{n=1}^{\infty} \frac{\rho^n}{(\rho+1)\cdots(\rho+n)} = \Gamma(\rho+1)\sum_{n=0}^{\infty} \frac{\rho^n}{\Gamma(\rho+n+1)} = \rho^{1-\rho} e^{\rho} \gamma(\rho,\rho), \qquad (3.109)$$

with $\gamma(a,x)$ the incomplete gamma function; see (3.65) and [Abramowitz&Stegun 1965], 6.5.4 and 6.5.29.

It follows from (3.108), (3.105) and (3.106) that

$$f_i = \frac{1}{\lambda}\sum_{n=i}^{\infty} \frac{\rho^{n-i+1}}{(\rho+i)\cdots(\rho+n)} \quad (i = 0,1,\ldots); \qquad (3.110)$$

$$g_i = \sum_{n=i}^{\infty} \frac{\rho^{n-i+1}}{(\rho+i)\cdots(\rho+n)} \quad (i = 0,1,\ldots). \qquad (3.111)$$

Finally, using (3.106), we can rewrite (3.102) as

$$\begin{aligned}
g_{\text{ITD}}^{\text{exp}}(K) &= \frac{a_B + b_B(K+g_K) + b_I(\lambda E\{T_K^{\text{exp}}\} - K)}{E\{T_K^{\text{exp}}\} + \frac{1}{\lambda}g_K} \\
&= \lambda b_I - \frac{(b_I - b_B)(K+g_K) - a_B}{E\{T_K^{\text{exp}}\} + \frac{1}{\lambda}g_K}, \qquad (3.112)
\end{aligned}$$

where $E\{T_K^{\text{exp}}\}$ is given by (3.99) and g_K by (3.111) with $i = K$. For computational purposes it is better to use (3.109) and (3.105) instead of (3.111).

3.5.2 An Erlang-n distributed delay-limit

If the delay-limit has an Erlang-n distribution, then the waiting customers pass through n exponentially distributed stages before their delay-limit expires. This leads to a state-space

$$\Omega := \{\mathbf{r} := (r_1,\ldots,r_n) \mid r_i \in I\!N,\ i = 1,\ldots,n\}, \qquad (3.113)$$

with r_i the number of customers with a residual delay-limit of i phases. A policy for the Erlang-n model is a function $\pi : \Omega \to \{0,1\}$, where $\pi(\mathbf{r}) = 1$ if a batch service is started in state $\mathbf{r}$ and $\pi(\mathbf{r}) = 0$ else. To find an optimal policy, we can formulate a SMDP with state space (3.113). Although this SMDP is similar to the MDP of section 2.6, there are some important differences:

- the delay-limit consists of exponentially distributed phases;
- the transition times are stochastic and state-dependent;
- events (arrivals and phase completions) occur one at a time.

The optimal policy

If the process is in state $\mathbf{r}$, then the expected time until the next event is $\frac{1}{\lambda+r_{1,n}\mu}$ (where $r_{1,n} := \sum_{i=1}^{n} r_i$). Moreover, the next event is a customer arrival with probability $\frac{\lambda}{\lambda+r_{1,n}\mu}$, a delay-limit expiring with probability $\frac{r_1\mu}{\lambda+r_{1,n}\mu}$, and a customer moving from phase i to phase $i-1$ with probability $\frac{r_i\mu}{\lambda+r_{1,n}\mu}$ ($i = 2,\ldots,n$). It follows that the optimality equations are given by

$$\begin{aligned} v(\mathbf{r}) \;=\; & \min\Big\{a_B + b_B r_{1,n} - \frac{g}{\lambda} + v(\mathbf{e}_n),\ \frac{1}{\lambda + r_{1,n}\mu}\Big(r_1\mu\, b_I - g + \\ & \lambda\, v(\mathbf{r}+\mathbf{e}_n) + r_1\mu\, v(\mathbf{r}-\mathbf{e}_1) + \sum_{i=2}^{n} r_i\mu\, v(\mathbf{r}+\mathbf{e}_{i-1}-\mathbf{e}_i)\Big)\Big\} \quad (\mathbf{r} \in \Omega). \end{aligned} \tag{3.114}$$

However, we need to be careful here. Since the delay-limit is stochastic, the epochs at which a delay-limit expires are not known in advance. This raises the following question: is it possible to include a customer in a batch service when his delay-limit expires? In other words, is it possible to postpone a batch service until a delay-limit expires, as is done under the IGCG-policy? The optimality equations (3.114) implicitly assume that this is *not* possible, i.e, when a delay-limit expires it is not possible to start a batch service that includes the customer whose delay-limit expires. On the other hand, the ITD-policy of the previous section assumes that it is possible to wait for a delay-limit to expire. Therefore it is necessary to add an auxiliary 0-1 variable to (3.113) that indicates whether or not a delay-limit is about to expire. Let $(\mathbf{r},1)$ denote the states where a delay-limit is about to expire and $(\mathbf{r},0)$ the other states. Obviously, a batch service is only started at epochs that a delay-limit is about to expire, so that the optimality equations now become

$$\begin{aligned} v(\mathbf{r},0) \;=\; & \frac{1}{\lambda + r_{1,n}\mu}\Big(-g + \lambda\, v(\mathbf{r}+\mathbf{e}_n,0) + r_1\mu\, v(\mathbf{r},1) + \sum_{i=2}^{n} r_i\mu\, v(\mathbf{r}+\mathbf{e}_{i-1}-\mathbf{e}_i,0)\Big); \\ v(\mathbf{r},1) \;=\; & \min\Big\{a_B + b_B r_{1,n} - \frac{g}{\lambda} + v(\mathbf{e}_n),\ \frac{1}{\lambda + r_{1,n}\mu}\Big(b_I - g + \\ & \lambda\, v(\mathbf{r}-\mathbf{e}_1+\mathbf{e}_n,0) + (r_1-1)\mu\, v(\mathbf{r}-\mathbf{e}_1,1) + \sum_{i=2}^{n} r_i\mu\, v(\mathbf{r}-\mathbf{e}_1+\mathbf{e}_{i-1}-\mathbf{e}_i,0)\Big)\Big\} \end{aligned} \tag{3.115}$$

for $\mathbf{r} \in \Omega$.

Analysis of a fixed policy

A fixed policy π for the Erlang-n model is completely characterized by the set of states $\Pi \subset \Omega$ where a batch service is started, i.e.,

$$\pi(\mathbf{r}) = \begin{cases} 1 & \text{if } \mathbf{r} \in \Pi \\ 0 & \text{else} \end{cases} \quad \Longleftrightarrow \quad \Pi = \{\mathbf{r} \mid \pi(\mathbf{r}) = 1\}. \tag{3.116}$$

We now describe a general method to compute the costs of any heuristic policy, which is similar to the "brute-force" method described in Appendix 2.B. Consider a given policy characterized by the set Π, and define

$$\begin{aligned}
g_\Pi &:= \text{expected average costs for the policy characterized by the set } \Pi; \\
s_\Pi(\mathbf{r}) &:= \text{expected first entrance time into } \Pi \text{ starting in state } \mathbf{r} \quad (\mathbf{r} \in \Omega); \\
y_\Pi(\mathbf{r}) &:= \text{expected number of individual services until the first entrance into } \Pi \\
&\qquad \text{starting in state } \mathbf{r} \quad (\mathbf{r} \in \Omega); \\
z_\Pi(\mathbf{r}) &:= \text{expected number of customers included in the batch upon first entrance} \\
&\qquad \text{into } \Pi \text{ starting in state } \mathbf{r} \quad (\mathbf{r} \in \Omega).
\end{aligned}$$

Using (3.4), it is easily verified that

$$g_\Pi = \frac{a_B + b_B z_\Pi(\mathbf{0}) + b_I y_\Pi(\mathbf{0})}{s_\Pi(\mathbf{0})}. \tag{3.117}$$

The required quantities $s_\Pi(\mathbf{0})$, $y_\Pi(\mathbf{0})$ and $z_\Pi(\mathbf{0})$ can all three be computed by solving a linear system of size $|\Omega - \Pi|$:

$$\begin{aligned}
s_\Pi(\mathbf{r}) &= \frac{1}{\lambda + r_{1,n}\mu}\left(1 + \lambda\, s_\Pi(\mathbf{r}{+}\mathbf{e}_n) + r_1\mu\, s_\Pi(\mathbf{r}{-}\mathbf{e}_1) + \sum_{i=2}^{n} r_i\mu\, s_\Pi(\mathbf{r}{+}\mathbf{e}_{i-1}{-}\mathbf{e}_i)\right) \quad (\mathbf{r} \notin \Pi); \\
s_\Pi(\mathbf{r}) &= 0 \quad (\mathbf{r} \in \Pi),
\end{aligned} \tag{3.118}$$

$$\begin{aligned}
y_\Pi(\mathbf{r}) &= \frac{1}{\lambda + r_{1,n}\mu}\left(\lambda\, y_\Pi(\mathbf{r}{+}\mathbf{e}_n) + r_1\mu\left(1 + y_\Pi(\mathbf{r}{-}\mathbf{e}_1)\right) + \sum_{i=2}^{n} r_i\mu\, y_\Pi(\mathbf{r}{+}\mathbf{e}_{i-1}{-}\mathbf{e}_i)\right) \quad (\mathbf{r} \notin \Pi); \\
y_\Pi(\mathbf{r}) &= 0 \quad (\mathbf{r} \in \Pi),
\end{aligned} \tag{3.119}$$

$$\begin{aligned}
z_\Pi(\mathbf{r}) &= \frac{1}{\lambda + r_{1,n}\mu}\left(\lambda\, z_\Pi(\mathbf{r}{+}\mathbf{e}_n) + r_1\mu\, z_\Pi(\mathbf{r}{-}\mathbf{e}_1) + \sum_{i=2}^{n} r_i\mu\, z_\Pi(\mathbf{r}{+}\mathbf{e}_{i-1}{-}\mathbf{e}_i)\right) \quad (\mathbf{r} \notin \Pi); \\
z_\Pi(\mathbf{r}) &= r_{1,n} \quad (\mathbf{r} \in \Pi).
\end{aligned} \tag{3.120}$$

Since the state space of the Erlang-n model is equivalent to that of the discrete-time model, we can use similar policies as in the previous chapter. The discrete-time Critical-Group policy corresponds to $\Pi = \{\mathbf{r} \mid r_1 \geq K\}$, the Total-Demand policy to $\Pi = \{\mathbf{r} \mid r_{1,n} \geq K\}$ and the Extended Total-Demand policy to $\Pi = \{\mathbf{r} \mid r_{1,n} \geq K_1 \text{ and } r_1 \geq K_2\}$.

3.5.3 Numerical results

We conclude this section with some computational results for the model with an exponential delay-limit of subsection 3.5.1. In Table 3.4 we give $g_{\text{TD}}(K)$ (from (3.95)) and $g_{\text{ITD}}(K)$ (from (3.112)) for $\mu = 1$, $\lambda \in \{2, 5, 10, 20\}$ and some appropriate values of K. When comparing Table 3.4 to Table 3.3, we see that the expected costs for an exponential delay-limit are considerably higher than for a constant delay-limit. This is in accordance with the general phenomenon that more uncertainty leads to higher costs.

λ	K	$E\{S_{\mathrm{TD}}^{\exp}\}$	$E\{Y_{\mathrm{TD}}^{\exp}\}$	f_K	g_K	$g_{\mathrm{TD}}^{\exp}(K)$	$g_{\mathrm{ITD}}^{\exp}(K)$
2	1	0.5	0	0.5973	1.1945	4	1.8227
	2	1.2500	0.5000	0.3959	0.7918	2.0000	1.5189
	3	2.5000	2.0000	0.2918	0.5836	1.6000	1.4328
	4	4.8750	5.7500	0.2295	0.4590	1.5897	1.5183
	5	10.1250	15.2500	0.1884	0.3769	1.7037	1.6726
5	1	0.2	0	0.4377	2.1887	25	7.8403
	2	0.4400	0.2000	0.3253	1.6264	11.8182	6.7949
	3	0.7360	0.6800	0.2554	1.2770	7.7174	5.7293
	4	1.1136	1.5680	0.2086	1.0431	5.8980	4.9674
	5	1.6157	3.0784	0.1755	0.8776	5.0000	4.5100
	6	2.3178	5.5888	0.1511	0.7553	4.5685	4.2890
	7	3.3603	9.8013	0.1323	0.6616	4.4048	4.2379
	8	5.0198	17.0988	0.1176	0.5879	4.4024	4.3016
	9	7.8749	30.3747	0.1057	0.5284	4.4921	4.4326
	10	13.2143	56.0714	0.0959	0.4796	4.6216	4.5883
10	1	0.1	0	0.3333	3.3327	100	23.0800
	2	0.2100	0.1000	0.2666	2.6660	48.0952	21.1917
	3	0.3320	0.3200	0.2199	2.1992	31.0843	18.6983
	4	0.4686	0.6860	0.1859	1.8590	22.8041	16.3270
	5	0.6232	1.2324	0.1603	1.6026	18.0226	14.3362
	6	0.8006	2.0056	0.1404	1.4039	14.9965	12.7590
	7	1.0070	3.0695	0.1246	1.2462	12.9793	11.5499
	8	1.2514	4.5143	0.1119	1.1186	11.5982	10.6465
	9	1.5470	6.4701	0.1013	1.0135	10.6464	9.9918
	10	1.9130	9.1303	0.0926	0.9256	10.0000	9.5385
	11	2.3790	12.7905	0.0851	0.8511	9.5797	9.2488
	12	2.9917	17.9167	0.0787	0.7874	9.3315	9.0922
	13	3.8268	25.2682	0.0732	0.7322	9.2161	9.0430
	14	5.0125	36.1251	0.0684	0.6840	9.2020	9.0781
	15	6.7725	52.7248	0.0642	0.6416	9.2617	9.1748
20	1	0.05	0	0.2482	4.9632	400	67.0781
	5	0.2777	0.5539	0.1406	2.8128	74.0161	49.1324
	10	0.6527	3.0547	0.0872	1.7433	35.3201	31.1591
	15	1.2148	9.2952	0.0621	1.2415	24.1160	22.9436
	20	2.2237	24.4736	0.0479	0.9577	20.0000	19.5784
	21	2.5384	29.7672	0.0458	0.9154	19.6060	19.2588
	22	2.9188	36.3754	0.0438	0.8766	19.3148	19.0290
	23	3.3872	44.7445	0.0420	0.8409	19.1143	18.8800
	24	3.9759	55.5190	0.0404	0.8079	18.9940	18.8029
	25	4.7324	69.6483	0.0389	0.7773	18.9435	18.7892
	26	5.7280	88.5600	0.0374	0.7489	18.9525	18.8294
	27	7.0723	114.4452	0.0361	0.7225	19.0102	18.9136
	28	8.9370	150.7402	0.0349	0.6978	19.1048	19.0305
	29	11.5977	202.9532	0.0337	0.6748	19.2240	19.1682
	30	15.5056	280.1120	0.0327	0.6531	19.3551	19.3144

Table 3.4: $g_{\mathrm{TD}}^{\exp}$ and $g_{\mathrm{ITD}}^{\exp}$ for $\mu = 1$ ($\lambda = 2, 5, 10, 20$)

λ	a_B	g_{NB}	g_{OB}	$g_{\text{TD}}(K^*)$	g_{GCG} $(K^*;C^*)$	$\hat{g}_{\text{IGCG}}$ $(K^*;C^*)$	$g^{\text{exp}}_{\text{TD}}(K^*)$	$g^{\text{exp}}_{\text{ITD}}(K^*)$
1	1.5	1	0.5	0.5081 (3)	0.5072 (3;1.9209)	0.4774 (2;1.9527)	0.7 (3)	0.6076 (2)
	2	1	0.6667	0.6187 (4)	0.6187 (4;2)	0.6148 (2;1.3734)	0.7949 (4)	0.7164 (3)
	2.5	1	0.8333	0.7140 (4)	0.7140 (4;2)	0.7185 (2;1.9683)	0.8462 (4)	0.8059 (3)
3	4.5	3	1.9286	1.9231 (8)	1.9231 (8;2)	1.8489 (3;0.8360)	2.4569 (8)	2.3304 (7)
	6	3	2.5714	2.3292 (9)	2.3292 (9;2)	2.3383 (5;1.2082)	2.6732 (9)	2.5959 (8)
	7.5	3	3.2143	2.6071 (10)	2.6071 (10;2)	2.6650 (8;1.7855)	2.8164 (10)	2.7733 (9)
5	7.5	5	3.4091	3.3936 (12)	3.3936 (12;2)	3.2978 (3;0.4535)	4.2479 (12)	4.1171 (11)
	10	5	4.5455	4.1306 (14)	4.1306 (14;2)	4.1745 (7;1.0229)	4.6010 (14)	4.5215 (13)
	12.5	5	5.6818	4.5817 (16)	4.5817 (16;2)	4.6716(14;1.9827)	4.8154 (15)	4.7702 (15)
10	15	10	7.1429	7.0802 (21)	7.0802 (21;2)	6.9917 (4;0.3156)	8.8009 (22)	8.6619 (21)
	20	10	9.5238	8.7644 (25)	8.7642(25;1.9932)	8.8930(11;0.8227)	9.4717 (25)	9.3946 (25)
	25	10	11.9048	9.5897 (29)	9.5897 (29;2)	9.7237(27;1.9739)	9.8276 (29)	9.7939 (28)

Table 3.5: Numerical comparison of different policies for $D = 2$

λ	a_B	g_{NB}	g_{OB}	$g_{\text{TD}}(K^*)$	g_{GCG} $(K^*;C^*)$	$\hat{g}_{\text{IGCG}}$ $(K^*;C^*)$	$g^{\text{exp}}_{\text{TD}}(K^*)$	$g^{\text{exp}}_{\text{ITD}}(K^*)$
1	2.25	1	0.5625	0.5348 (5)	0.5344 (5;2.9485)	0.5367 (3;3)	0.7539 (4)	0.6869 (4)
	3	1	0.75	0.6558 (6)	0.6545 (5;2.7436)	0.6806 (3;2.1578)	0.8328 (5)	0.7859 (4)
	3.75	1	0.9375	0.7419 (6)	0.7419 (6;3)	0.7930 (4;2.5240)	0.8931 (6)	0.8561 (5)
3	6.75	3	2.025	2.0170 (12)	1.9873 (9;2.3266)	1.9550 (3;0.7700)	2.5315 (11)	2.4449 (10)
	9	3	2.7	2.3958 (13)	2.3949(13;2.9560)	2.4738 (6;1.4074)	2.7464 (12)	2.6924 (12)
	11.25	3	3.375	2.6665 (15)	2.6665 (15;3)	2.7789 (13;3)	2.8748 (14)	2.8481 (14)
5	11.25	5	3.5156	3.5457 (18)	3.4727(14;2.3839)	3.4257 (4;0.6514)	4.3445 (17)	4.2544 (16)
	15	5	4.6875	4.2220 (20)	4.2191(20;2.9427)	4.3464 (9;1.3255)	4.6875 (20)	4.6341 (19)
	18.75	5	5.8594	4.6520 (23)	4.6520 (23;3)	4.7970 (21;3)	4.8784 (22)	4.8543 (22)
10	22.5	10	7.2581	7.3995 (32)		7.1445 (4;0.3048)	8.9284 (32)	8.8315 (31)
	30	10	9.6774	8.8937 (38)		9.1231(16;1.2534)	9.5845 (37)	9.5331 (36)
	37.5	10	12.0968			9.8501(40;2.9811)	9.9008 (42)	9.8822 (41)

Table 3.6: Numerical comparison of different policies for $D = 3$

3.6 Numerical comparisons

In this section we compare all policies and approximations discussed in this chapter, by computing the optimal parameters for the respective policies for a range of values of λ, D and a_B. In order to compare the continuous-time (CT) policies of this chapter to the discrete-time (DT) policies of the previous chapter, we use the same parameter settings as in section 2.7. Note that the use of the parameter λ is consistent with section 2.7, since under a Poisson arrival process with rate λ the number of customers per period of length 1 has a Poisson(λ) distribution. The DT policies assume that a batch service can only be started at the end of a period, while under the CT policies a batch service can be started at any time. Table 3.5 gives the minimal expected costs for the NB- (from (3.5)), OB- (from (3.10)), TD- (from (3.23)), GCG- (from (3.28)) and IGCG-policy (from (3.89)), as well as for the exponential TD- (from (3.95)) and ITD-policy (from (3.112)), for $D = 2$. We use the parameter settings $\lambda \in \{1, 3, 5, 10\}$, $a_B \in \{0.75\lambda D, \lambda D, 1.25\lambda D\}$, $b_B = 0$ and $b_I = 1$ (just like in Table 2.1). Table 3.6 repeats the computations for $D = 3$ (see also Table 2.2); the four empty cells are due to computational infeasibility. Recall that for $a_B \ll \lambda D$

($a_B \gg \lambda D$) the OB-policy (NB-policy) is a good policy to use, as none of the other policies perform significantly better in these cases. Also, the optimal control parameters for the CT policies only depend on λ and D via λD.

Some conclusions of interest can be drawn from these results. First of all, when comparing Tables 3.5 and 3.6 to Tables 2.1 and 2.2, we see that the CT TD-, GCG- and IGCG-policies do better than the optimal DT policy for all instances considered here, although the difference in costs decreases with λD. For $D = 2$, $\lambda = 1$ and $a_B \in \{1.5, 2\}$ even the CT OB-policy is better than the DT optimal policy. Obviously, the lower the mean number of customers between consecutive decision epochs (λ), the more restrictive the assumption of fixed decision epochs and the higher the gain of CT policies with respect to DT policies. Due to the infeasibility of the optimal policy (see section 3.1), the question remains how much the CT heuristic policies lose with respect to the CT optimal policy.

For relatively low values of a_B (see the lines with $a_B = 0.75\lambda D$ in Tables 3.5 and 3.6) the GCG-policy only slightly improves upon the OB-policy (if at all), but for higher values of a_B the TD- and GCG-policy perform considerably better. It is remarkable that for $D = 2$ the optimal GCG-policy always coincides with the optimal TD-policy (i.e., $C^* = D$), except for $\lambda=1, a_B=1.5$ and $\lambda=10, a_B=20$. Although it occurs more often that $C^* < D$ for $D = 3$, it is clear that the GCG-policy does not add much to the TD-policy. Moreover, the computational effort required to find the optimal (K, C) pair for the GCG-policy (using the bisection algorithm described in section 3.4.2) increases heavily with K; for example, for $\lambda = 5, a_B = 15$ and an accuracy of 10^{-4} the computation time is already in the order of hours on a Pentium PC. This is due to the fact that the computation time of $\hat{E}\{T_K^{(C)}\}$ (see (3.73)) increases exponentially with K (see (3.68)–(3.69), Theorem 3.3 and Table 3.2).

Due to a larger degree of randomness, the embedded Markov chain approximation for the IGCG-policy is only accurate for small values of K and clearly overestimates the true costs (see section 3.4). This explains why the approximated costs of the IGCG-policy are mostly higher than the (approximated) costs of the GCG-policy. Also, both K^* and C^* are smaller than for the GCG-policy, while only a few instances have $C^* = D$. As noted earlier, the approximation is very efficient in terms of computation time.

When the delay-limit is exponentially distributed, the residual delay-limit is also exponentially distributed and the only relevant policies are the TD- and ITD-policy (see section 3.5). Obviously, the ITD-policy is always better than the TD-policy, where the difference in costs decreases with λ and a_B. The expected average costs for an exponential delay-limit are considerably higher than for a constant delay-limit, implying that the service model is very sensitive to the delay-limit distribution. In general, one may expect that the higher the coefficient of variation ("randomness") of the delay-limit, the higher the expected average costs of the model (this is not completely trivial; see [Ridder et al. 1996] for a counterexample in the context of the "newsboy" inventory model). In this respect the case of a constant delay-limit and the case of an exponentially distributed delay-limit are two extreme cases, whereas the embedded Markov chain approximation is somewhere in between.

Appendix 3.A: Proof of Theorem 3.2

In this appendix we prove Theorem 3.2, stating that every sample path of the process $\{\mathbf{X}_k\}$ (defined in section 3.2) has equal probability. In order to do so, we need the following results.

Lemma 3.3 *Let $\{U_{ij};\ j = 1,\ldots,j_i\}$ $(i = 1,\ldots,n)$ be n finite sequences of mutually independent and identically distributed random variables with a uniform distribution over (a,b). Then ($j_{1,n} := \sum_{i=1}^{j_i}$)*

(i) $\Pr\{U_{rs} = \min_{\substack{i=1,\ldots,n;\\ j=1,\ldots,j_i}} U_{ij}\} = \dfrac{1}{j_{1,n}} \quad (r = 1,\ldots,n;\ s = 1,\ldots,j_r);$

(ii) $\Pr\{\min_{\substack{i=1,\ldots,n;\\ j=1,\ldots,j_i}} U_{ij} = \min_{j=1,\ldots,j_r} U_{rj}\} = \dfrac{j_r}{j_{1,n}} \quad (r = 1,\ldots,n).$

Proof. (i) Define

$$\begin{aligned} E_{rs} &:= \{U_{rs} = \min_{\substack{i=1,\ldots,n;\\ j=1,\ldots,j_i}} U_{ij}\} \quad (r = 1,\ldots,n;\ s = 1,\ldots,j_r); \\ E_r &:= \{\min_{\substack{i=1,\ldots,n;\\ j=1,\ldots,j_i}} U_{ij} = \min_{j=1,\ldots,j_r} U_{rj}\} \quad (r = 1,\ldots,n). \end{aligned}$$

Conditioning on U_{rs} yields

$$\begin{aligned} \Pr\{E_{rs}\} &= \int_a^b \frac{1}{b-a} \Pr\{U_{ij} \geq x,\ i = 1,\ldots,n,\ j = 1,\ldots,j_i \mid U_{rs} = x\}\, dx \\ &= \int_a^b \frac{1}{b-a} \left(\frac{b-x}{b-a}\right)^{j_{1,n}-1} dx \\ &= \frac{1}{j_{1,n}} \quad (r = 1,\ldots,n;\ s = 1,\ldots,j_r). \end{aligned}$$

(ii) Since $E_r = \bigcup_{s=1}^{j_r} E_{rs}$ and $\bigcap_{s=1}^{j_r} E_{rs} = \emptyset$, it follows that

$$\Pr\{E_r\} = \Pr\{\bigcup_{s=1}^{j_r} E_{rs}\} = \sum_{s=1}^{j_r} \Pr\{E_{rs}\} = \frac{j_r}{j_{1,n}} \quad (r = 1,\ldots,n). \quad \square$$

Lemma 3.4 *For $x_i \leq k_i$ $(i = 1,\ldots,n)$ with $\sum_{i=1}^n x_i = k$, and any $t \geq 0$, we have that*

$$\Pr\{\mathbf{X}_{k+1} = \mathbf{x} + \mathbf{e}_r \mid \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} = \frac{k_r - x_r}{\sum_{i=1}^n (k_i - x_i)} \quad (r = 1,\ldots,n).$$

Proof. Define $f(x) := \frac{d}{dx}\Pr\{R_k \leq x \mid \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\}$. By conditioning with respect to $f(x)$ and using Lemma 3.3 it follows that

$$\begin{aligned} &\Pr\{\mathbf{X}_{k+1} = \mathbf{x} + \mathbf{e}_r \mid \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} \\ =\ & \int_0^t f(u) \Pr\{\mathbf{X}_{k+1} = \mathbf{x} + \mathbf{e}_r \mid R_k = u;\ \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} \\ =\ & \int_0^t f(u) \Pr\{\min_{\substack{i=1,\ldots,n;\\ j=x_i+1,\ldots,k_i}} R_{ij} = \min_{j=x_r+1,\ldots,k_r} R_{rj} \mid R_k = u;\ \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} \\ =\ & \frac{k_r - x_r}{\sum_{i=1}^n (k_i - x_i)}, \end{aligned}$$

since

$$\Pr\{R_{ij} \leq x \mid R_k = u;\ \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} = \Pr\{U \leq x\} \quad (u \leq x \leq t)$$

with U uniformly distributed over $[u, t]$. □

Lemma 3.5 *For $x_i \leq k_i$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^n x_i = k$, and any $t \geq 0$, we have that*

$$\Pr\{\mathbf{X}_k = \mathbf{x} \mid \mathbf{N}(t) = \mathbf{k}\} \;=\; \frac{\prod_{i=1}^n \binom{k_i}{x_i}}{\binom{k_{1;n}}{k}}.$$

Proof. We use induction on k. For $k = 1$ the result reduces to

$$\Pr\{\mathbf{X}_1 = \mathbf{e}_r \mid \mathbf{N}(t) = \mathbf{k}\} = \frac{k_r}{k_{1;n}} \quad (r = 1, \ldots, n),$$

and this is true by Lemma 3.4 with $k = 0$ and $\mathbf{x} = \mathbf{0}$. Conditioning on $\mathbf{X}_{k-1}$ and then using the induction hypothesis together with Lemma 3.4 yields

$$\begin{aligned}
&\Pr\{\mathbf{X}_k = \mathbf{x} \mid \mathbf{N}(t) = \mathbf{k}\} \\
=\;& \sum_{r=1}^n \Pr\{\mathbf{X}_{k-1} = \mathbf{x} - \mathbf{e}_r \mid \mathbf{N}(t) = \mathbf{k}\} \Pr\{\mathbf{X}_k = \mathbf{x} \mid \mathbf{X}_{k-1} = \mathbf{x} - \mathbf{e}_r;\ \mathbf{N}(t) = \mathbf{k}\} \\
=\;& \sum_{r=1}^n \frac{\binom{k_r}{x_r - 1} \prod_{\substack{i=1 \\ i \neq r}}^n \binom{k_i}{x_i}}{\binom{k_{1;n}}{k-1}} \cdot \frac{k_r - x_r + 1}{\sum_{i=1}^n (k_i - x_i) + 1} \\
=\;& \sum_{r=1}^n \frac{x_r \prod_{i=1}^n \binom{k_i}{x_i}}{\binom{k_{1;n}}{k-1} (k_{1;n} - k + 1)} \;=\; \frac{\prod_{i=1}^n \binom{k_i}{x_i}}{\binom{k_{1;n}}{k}},
\end{aligned}$$

since $\sum_{r=1}^n x_r = k$. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2.
Since $\mathbf{x}_k - \mathbf{x}_{k-1} = \mathbf{e}_r$ if and only if the k^{th} event is an arrival in I_r it follows that

$$\begin{aligned}
&\Pr\{\mathbf{X}_k = \mathbf{x}_k;\ k = 1, \ldots, k_{1;n} \mid \mathbf{N}(t) = \mathbf{k}\} \\
=\;& \prod_{k=1}^{k_{1;n}} \Pr\{\mathbf{X}_k = \mathbf{x}_k \mid \mathbf{X}_{k-1} = \mathbf{x}_{k-1};\ \mathbf{N}(t) = \mathbf{k}\} \\
=\;& \prod_{k=1}^{k_{1;n}} \frac{\sum_{r=1}^n I_{\{\mathbf{x}_k - \mathbf{x}_{k-1} = \mathbf{e}_r\}} (k_r - x_{kr})}{\sum_{i=1}^n (k_i - x_{ki})},
\end{aligned} \tag{3.121}$$

where in the first equality we use the Markov property

$$\begin{aligned} & \Pr\{\mathbf{X}_k = \mathbf{x}_k \mid \mathbf{X}_1 = \mathbf{x}_1, \ldots, \mathbf{X}_{k-1} = \mathbf{x}_{k-1};\ \mathbf{N}(t) = \mathbf{k}\} \\ = \ & \Pr\{\mathbf{X}_k = \mathbf{x}_k \mid \mathbf{X}_{k-1} = \mathbf{x}_{k-1};\ \mathbf{N}(t) = \mathbf{k}\}. \end{aligned}$$

Now observe that in the numerator of (3.121) every factor $k_i - j$ ($j = 0, \ldots, k_i - 1$) occurs exactly once for all $i = 1, \ldots, n$, so that the product of these $k_{1;n}$ factors is just $k_1! \cdots k_n!$. The denominator of (3.121) obviously equals $(k_{1;n})!$, and hence the desired result follows. □

Appendix 3.B: Relation between Theorem 3.1 and 3.3

In this appendix we show the equivalence of Theorem 3.1(i) for $t = nC$ and Theorem 3.3 for $K = 2$. Substituting $K = 2$ in Theorem 3.3 we have that

$$\bar{F}_{T_2^{(C)}}(nC) = e^{-\lambda nC} \sum_{k=0}^{n} (\lambda C)^k \sum_{\substack{k_1,\ldots,k_n \in \{0,1\} \\ k_{1;n}=k}} \det(C_n(\mathbf{k})) \quad (n = 1, 2, \ldots), \tag{3.122}$$

with $C_n(\mathbf{k})$ given by (3.53). Here k_i denotes the number of arrivals in interval I_i and $k = k_{1;n}$ the total number of arrivals in $[0, nC]$. Now consider the string $(0, \mathbf{k}, 0)$, consisting of k ones and $n - k + 2$ zeros, and define

$$o_j \ := \ \text{number of ones between } (j-1)^{\text{th}} \text{ and } j^{\text{th}} \text{ zero} \quad (j = 1, \ldots, n-k+1)$$

and $\mathbf{o} := (o_1, \ldots, o_{n-k+1})$. For example, if $n = 8$, $k = 4$ and $\mathbf{k} = (0,1,0,1,1,0,0,1)$ then $\mathbf{o} = (0,1,2,0,1)$. Note that any string $\mathbf{l}$ is uniquely determined by $\mathbf{k}$ and vice versa. Now, if $\mathbf{k}$ is a string of k ones and $n - k$ zeros, it can be shown that

$$\det(C_n(\mathbf{k})) = \prod_{j=1}^{n-k+1} \det(C_{o_j}(\mathbf{1}_{o_j})) = \prod_{j=1}^{n-k+1} \frac{1}{o_j!} \quad (n \geq 1;\ 0 \leq k \leq n), \tag{3.123}$$

where $\mathbf{1}_n$ is a string of n ones. The first equality in (3.123) follows by observing that the terms with $k_i = 0$ can be left out in (3.45) and applying (3.52). To prove the second equality in (3.123), note that the matrix $C_n(\mathbf{1}_n)$ has the form

$$C_n := C_n(\mathbf{1}_n) = \begin{pmatrix} 1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{n!} \\ 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(n-1)!} \\ 0 & 1 & 1 & \cdots & \frac{1}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad (n \geq 1). \tag{3.124}$$

Expanding C_n in the cofactors of its first column iteratively and defining $\det(C_0) := 1$, we find that

$$\det(C_n) = \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i!} \det(C_{n-i}) \quad (n = 1, 2, \ldots). \tag{3.125}$$

It is easily verified that the solution of (3.125) is $\det(C_n) = \frac{1}{n!}$; indeed, substituting $\det(C_n) = \frac{1}{n!}$ reduces (3.125) to a special case of the binomium of Newton,

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0 \quad (n \geq 1). \tag{3.126}$$

Finally, applying the transformation $\mathbf{o}$ to (3.122) and using (3.123) and the multinomial identity

$$\sum_{\substack{o_1,\ldots,o_{n-k+1}\geq 0 \\ o_{1;n-k+1}=k}} \binom{k}{o_1,\ldots,o_{n-k+1}} = (n-k+1)^k, \tag{3.127}$$

we obtain

$$\begin{aligned}
\bar{F}_{T_2^{(C)}}(nC) &= e^{-\lambda nC} \sum_{k=0}^{n} (\lambda C)^k \sum_{\substack{o_1,\ldots,o_{n-k+1}\geq 0 \\ o_{1;n-k+1}=k}} \frac{1}{o_1!} \cdots \frac{1}{o_{n-k+1}!} \\
&= e^{-\lambda nC} \sum_{k=0}^{n} \frac{\big((n-k+1)\lambda C\big)^k}{k!} \quad (n = 1, 2, \ldots),
\end{aligned} \tag{3.128}$$

in accordance with Theorem 3.1(i) for $t = nC$.

Part II

Inventory models with a delay-limit on backorders

Chapter 4

A general framework

4.1 Introduction

An essential characteristic of the service model of Part I is that it is not possible to prepare demand (for goods or service) of customers in advance, i.e., to include customer orders in a batch before they arrive. This assumption is trivially satisfied when considering the provision of some physical service requiring the presence of the "customers" (e.g., running a bus or taxi service, or delivery of goods). In Part II we shift attention to the production setting, and consider a producer of consumer items who has to satisfy customer demand within the delay-limit. In this setting a batch service corresponds to a production run, and an individual service to demand being lost or satisfied in some other way (against a fixed cost per item). Now the restriction that customer demand cannot be prepared in advance is no longer valid, unless the produced goods are non-exchangeable, i.e., an item produced for a specific customer cannot be used for any other customer (but in that case the assumption that all customer orders can be aggregated into the same batch may be no longer valid). Clearly, if the produced items are exchangeable it may be worthwhile to produce more than necessary and to build up an inventory of finished goods in anticipation of future demand (due to economies of scale). Therefore we now extend the service model to the context of production to inventory of exchangeable items, which leads to a single-item production/inventory model with a time-limit on backorders.

Besides the absence of a finished-goods inventory, two other basic assumptions of the service model need to be reconsidered for the production/inventory model, namely

- The delay-limit does not include the service time;
- There are no capacity restrictions on the number of simultaneous batch services, the size of a batch and the number of simultaneous individual services.

The first assumption is not restrictive if the service time is negligible, or if the delay-limit can be adjusted for the batch-service time (see also subsection 1.3.1). To illustrate, suppose that the delay-limit equals D periods *including* the service time. Moreover, suppose that the batch-service time and the individual service time are both constant and equal to L and $\tilde{L}$ periods, respectively, with $\tilde{L} \leq L < D$ (if $L \geq D$ then it is impossible to do a batch service within the delay-limit, while $\tilde{L} > L$ leads to a conceptually different model). Then we can apply the service model by using the adjusted delay-limit $D-L$. It is important to note that this adjustment is only possible if the batch-service time is constant (or bounded by a constant) and smaller than D. In the production/inventory setting the batch-service time becomes the production lead time, and we will assume throughout that the delay-limit includes the production lead time.

The second assumption guarantees that both batch and individual services can be carried out at any moment in time. Regarding simultaneous batch services, it is never desirable to do a batch service within time D of the previous one, and hence it suffices to assume that the batch-service time never exceeds D (although the assumption that the size of a batch is not bounded is still needed). Conversely, if the batch-service time never exceeds L then the number of simultaneous batch services is at most $\lceil \frac{L}{D} \rceil$. Consequently, for the service model with an adjusted delay-limit $D-L$, the assumption $L \leq D-L$ (or $D \geq 2L$) guarantees that there is at most one batch service outstanding at any time under an optimal policy (see also Theorem 4.2 in section 4.4). As for the number of simultaneous individual services, this is also bounded for any reasonable policy, e.g., for the Critical-Group policy with parameter K the number of simultaneously started individual services never exceeds K. In a production/inventory model the production capacity is generally an important model characteristic that must be taken into account, and it is reflected in the maximal number of simultaneous production runs and the maximal number of items that can be included in one production run.

This brings us to another important difference between the service model and the production/inventory model. Obviously, the service model is only useful if the adjusted delay-limit $D-L$ is positive, or $D > L$, since otherwise a batch service cannot be carried out within the delay-limit and only individual services are possible. On the other hand, for a production/inventory model with production lead time L the assumption $D > L$ is not crucial, because customer demand can also be satisfied from stock on hand or from a production run that will be completed within the delay-limit. The reason for this is, of course, that in the service model customer demand cannot be prepared before arrival, while in the production/inventory model customer demand can be produced in advance (finished-goods inventory). More generally speaking, this boils down to the basic difference between a queueing model and an inventory model. In a queueing model the service activity (e.g., a batch service) can only be carried out *after* customer arrival and customers incur delay through waiting time and/or service time. In an inventory model, however, the service activity (e.g., production of an item) can be carried out already *before* customer arrival, generating "serviceable inventory" through which customers can be served without having to wait (customers only incur delay if serviceable inventory is zero).

As a result, two fundamentally different cases arise for the production/inventory model, namely $D > L$ and $D \leq L$. In the former case it is possible to postpone production until $D-L$ time units *after* demand arrival, while in the latter case production must have been started at latest $D-L$ time units *before* demand arrival. We illustrate this difference for a discrete-time model with a production lead time of L periods and at most one production run at any time, by considering four situations:

I On-hand inventory is zero and no production run is underway;

II On-hand inventory is zero and a production run is underway;

III On-hand inventory is positive and no production run is underway;

IV On-hand inventory is positive and a production run is underway.

- *The case $D \leq L$.*
If $D \leq L$ then demand that arrives in situation I is automatically lost, because it takes D or more periods to execute a production run: "production to order" is not possible. Hence it is not useful to keep track of the number of waiting demands or their residual waiting times, and the optimal policy depends on on-hand inventory only. Furthermore, when situation I occurs a new run will be started either immediately or never. This case also includes the special case $D = 0$, corresponding to a lost-sales inventory model with constant lead times and at most one order outstanding ($D = 0 \leq L$ then trivially holds).

- *The case $D > L$.*
In this case production to order is possible. In situation I a queue of waiting demand may build up, but demand with a delay of more than $D - L$ periods is lost (because the production takes L periods). Just as with the service model, the optimal policy now depends on the residual delay-limits: the state vector in situation I is the $(D-L)$-dimensional vector $(r_0, \ldots, r_{D-L-1})$, with r_i denoting the number of demands with a residual delay-limit of i periods ($i = 0, \ldots, D–L–1$). In situation II no new production batch can be started, and depending on the demand during the production lead time the process moves to situation I or III when the batch is completed. In situation III arriving demand is satisfied from stock on hand, and it is clearly suboptimal to initiate a new run (assuming unlimited batch capacity). As a result, situation IV will not occur.

The relationship between the service model and the production/inventory model is summarized in Table 4.1, that gives the prevailing aspect (queueing and/or inventory) and the situations that may occur (I-IV above).

model/case	$D \leq L$	$D > L$
service model	*irrelevant*	queueing (I,II)
production/inventory model	inventory (II,III,IV)	queueing/inventory (I,II,III)

Table 4.1: Relationship between service and production/inventory model

This chapter is organized as follows. In the next section we describe a general framework for single-item production/inventory models where customer demand must be satisfied within the delay-limit D (possibly zero). In section 4.3 we give an overview of various related models that have been studied in the literature, including a survey of lost-sales inventory models. In section 4.4 we prove two general results, while in section 4.5 we preview the specific models that we will focus on in the remainder of the thesis. Finally, as we will make regular use of discrete renewal equations, we present some results from the discrete counterpart of renewal theory in section 4.6.

4.2 Description of the framework

The general framework for single-item production/inventory models is described by the following characteristics.

1 *Delay-limit D*
The delay-limit D determines the maximal customer lead time, or the maximal time a customer is willing to wait. Demand not satisfied within the delay-limit D is lost, where "lost" must be interpreted in a broad sense and also includes the case where this demand is satisfied by alternative means (e.g., contracted out or given "individual service"). We assume throughout that D is a non-negative constant. The case $D = 0$ corresponds to a lost-sales inventory model, where demand that arrives when on-hand inventory is zero is lost. We will refer to the case $D > 0$ as "a delay-limit on backorders", since demand is backordered until time D and is lost thereafter.

2 *Review system*
The review system may be either periodic review or continuous review. Under continuous review (or transactions reporting) every change in the system is noted immediately and a production run can be started at any point in time. On the other hand, under periodic review the system is reviewed at fixed points in time, e.g., every T time units. If the costs of reviewing are taken into account, then the frequency T can be seen as an additional decision variable. Periodic review leads to a discrete-time model (setting $T = 1$ w.l.o.g.), continuous review to a continuous-time model.

3 *Demand process*
Demand may be discrete or continuous, and stochastic or deterministic; we are only concerned with discrete stochastic demand for produced items. The demand process may be discrete-time or continuous-time, depending on the review system. For periodic-review models we describe the demand process as a sequence of i.i.d. discrete random variables $\{X_n\}$ with

$$X_n \quad := \quad \text{total demand in } n^{\text{th}} \text{ review period} \quad (n = 1, 2, \ldots).$$

For continuous-review models we describe the demand process as a compound renewal process

$$Y(t) = \sum_{n=1}^{N(t)} X_n \quad (t \geq 0), \tag{4.1}$$

with

$$
\begin{aligned}
Y(t) &:= \text{ total demand in } [0,t] \quad (t \geq 0); \\
N(t) &:= \text{ number of customer arrivals in } [0,t] \quad (t \geq 0); \\
X_n &:= \text{ number of items demanded by } n^{\text{th}} \text{ customer} \quad (n = 1,2,\ldots).
\end{aligned}
$$

Note that the definition of X_n depends on the context; this will prove useful in Chapter 5. We also define

$$q_k := \Pr\{X_n = k\}, \quad Q_k := \sum_{j=0}^{k} q_j, \quad \bar{Q}_k = \sum_{j=k}^{\infty} q_j \quad (k = 0,1,\ldots), \quad \mu := E\{X_n\};$$

$$S_n := \sum_{i=1}^{n} X_i, \quad S_{m,n} := \sum_{i=m}^{n} X_i = S_n - S_{m-1} \quad (n = 1,2,\ldots;\ m = 1,\ldots,n);$$

$$q_k^{(n)} := \Pr\{S_n = k\}, \quad Q_k^{(n)} := \sum_{j=0}^{k} q_j^{(n)}, \quad \bar{Q}_k^{(n)} = \sum_{j=k}^{\infty} q_j^{(n)} \quad (k = 0,1,\ldots).$$

4 *Production lead time L*
The production lead time L determines the time lag between the start and the completion of a production run, or the duration of a production run (in an inventory model with an outside supplier, L is the order lead time). We assume that the entire production batch becomes serviceable inventory at the end of the production run and not earlier. This distinguishes our models from production/inventory models where produced items are continuously added to inventory according to a – possibly controllable – production rate (see e.g. [Doshi et al. 1978] and [de Kok 1985]). Depending on the specific model, L may be deterministic or stochastic, and may or may not depend on the size of the production batch. Unless explicitly stated otherwise, we will assume that L is deterministic and independent of the batch size.

5 *Maximal number of simultaneous production runs N*
It may or may not be possible to have more than one production run simultaneously. In case $N = 1$, a new production run cannot be started before the previous run is completed, analogously to an inventory model with at most one order outstanding. In case $N > 1$, we can think of N identical machines allowing to process N batches simultaneously (possibly all having a different size and completion time). In case $N = \infty$, there are no restrictions on the number of simultaneous production runs. In an inventory model where items are ordered at an outside supplier it is often true that $N = \infty$, whereas in a production/inventory model it is more likely that $N < \infty$.

6 *Maximal batch size M*
The maximal batch size M determines the maximal number of items that can be included in one production batch. With N identical machines, this gives a total production capacity of NM items. Again, in an inventory model with an outside supplier it is often reasonable to assume that $M = \infty$.

7 *Cost structure*
Costs may be associated with production (ordering), holding, waiting and lost sales. In general terms, we define

$$
\begin{aligned}
c(i) &:= \text{production costs for a production batch of } i \text{ items;}\\
h(i) &:= \text{holding costs per unit of time when serviceable inventory is } i\text{;}\\
w(i) &:= \text{waiting costs per unit of time when } i \text{ customers are waiting;}\\
p(i) &:= \text{penalty costs for } i \text{ items lost.}
\end{aligned}
$$

We will assume throughout that

$$c(i) = K + ci, \quad h(i) = hi, \quad w(i) = 0, \quad p(i) = pi \quad (i = 0, 1, \ldots). \tag{4.2}$$

4.3 Related literature

4.3.1 Lost-sales inventory models: A survey

As noted earlier, the case $D = 0$, $N = \infty$ and $K = \infty$ corresponds to a single-item lost-sales inventory model with order lead time L. Since the pioneering work of [Arrow et al. 1958], a lot of research has been done on single-item dynamic inventory models; for a survey, see e.g. [Graves et al. 1993]. The vast majority of this research is devoted to the case where excess demand is completely backordered, while relatively few papers have been concerned with the case where excess demand is lost. The main reason for this is the complexity of the dynamic programming formulation for the lost-sales model with positive lead times. We will first illustrate the difficulties involved with the lost-sales assumption, and then review the literature on lead-time lost-sales models. For the sake of coherency we discuss the periodic-review case and the continuous-review case separately.

Periodic-review models

Consider a periodic-review model with an order lead time of $L > 0$ periods, ordering costs $c(i)$, holding costs $h(i)$ and penalty costs $p(i)$. If excess demand is completely backordered, the optimal ordering decision only depends on the inventory position, defined as stock on hand plus on order minus backorders (see [Arrow et al. 1958], Chapter 10, Theorem 1). Thus, the dynamic programming equations for the discounted-cost version of this problem can be written as

$$v(i) = \min_{a=0,1,\ldots} \left\{ c(a) + \alpha^{L+1} \sum_{k=0}^{\infty} q_k^{(L)} l(i-k+a) + \alpha \sum_{k=0}^{\infty} q_k v(i-k+a) \right\} \quad (i \in \boldsymbol{Z}), \tag{4.3}$$

where

$$l(i) := \begin{cases} \sum_{k=0}^{\infty} q_k \, p(k-i) & \text{if } i < 0; \\ \sum_{k=0}^{i} q_k \, h(i-k) + \sum_{k=i+1}^{\infty} q_k \, p(k-i) & \text{if } i \geq 0 \end{cases} \tag{4.4}$$

denotes the one-period holding and shortage costs starting with an inventory of i items (note that $p(i)$ denotes the shortage costs *per unit of time* when the inventory position is i). It is well-known that under mild conditions the optimal policy for this model is of the (s, S) type, i.e., order up to S whenever the inventory position drops to or below s (see [Iglehart 1963], [Veinott 1966] and [Zheng 1991]).

On the other hand, if all excess demand is lost then it is impossible to formulate a similar dynamic program with the inventory position as state variable (unless $L = 0$ or $L = 1$). This is due to the fact that the state transitions no longer solely depend on the inventory position, but on inventory on hand and on order *separately*, i.e., the state of the system cannot be "summarized" through the inventory position (see e.g. [Arrow et al. 1958], p. 157; [Wagner 1962], section 2.3.1; [Graves et al. 1993], p. 28). Consequently, the optimal ordering decision is a complex function of stock on hand i and the vector of outstanding orders $(j_1, \ldots, j_{L-1})$, with j_n the order quantity that will arrive in exactly n periods. The discounted dynamic programming equations now become

$$v(i, j_1, \ldots, j_{L-1}) = \min_{a=0,1,\ldots} \Big\{c(a) + l(i) + \alpha\Big(\sum_{k=0}^{i-1} q_k v(i-k+j_1, j_2, \ldots, a) + \bar{Q}_i v(j_1, \ldots, a)\Big)\Big\}. \quad (4.5)$$

In general, the optimal order quantities $a^*(i, j_1, \ldots, j_{L-1})$ do not exhibit a specific structure. In fact, we will show in Chapter 5 that even for the simple case $L = 1$ the function $a^*(i)$ may be non-monotone. Due to the complexity of the optimal policy, a large part of the literature on lost-sales models focuses on simple-structured policies with the additional stipulation that at most one order can be outstanding at any time ($N = 1$), since this reduces the state space to one dimension (on-hand inventory).

Most of the early work on lost-sales inventory models is devoted to the periodic-review continuous-demand model with no set-up costs ($K = 0$). For the case $L = 1$, where the state variable is on-hand inventory x (continuous), Karlin and Scarf prove that an optimal policy $a^*(x)$ is continuous and of the form

$$a^*(x) \begin{cases} > 0 & \text{if } x \leq s; \\ = 0 & \text{if } x > s, \end{cases} \quad (4.6)$$

with $a^*(x)$ strictly decreasing in x and $x + a^*(x)$ strictly increasing in x for $x \leq s$ (see [Arrow et al. 1958], 10.3). Morton ([Morton 1969a], [Morton 1969b]) uses the optimality equation to derive bounds for $a^*(x)$, $v(x)$ and $v'(x)$. His upper bound on $a^*(x)$ corresponds to a myopic policy: given on-hand inventory x and incoming orders $y_1, \ldots, y_{L-1}$ at the start of period 1 choose the order quantity y_L that minimizes ordering costs plus holding and penalty costs in period $L + 1$ (the holding and penalty costs in periods $1, \ldots, L$ are "sunk costs", i.e., they cannot be influenced by the order quantity). Interestingly, this problem leads to a "newsboy" equation (for a description of the newsboy model see e.g. [Graves et al. 1993], section 4.1):

$$P_1(x, y_1, \ldots, y_L) = \frac{p}{p + h}, \quad (4.7)$$

where $P_1(x, y_1, \ldots, y_L)$ is the probability of a stockout in period L+1 given the state vector $(x, y_1, \ldots, y_{L-1})$ and the order quantity y_L. A similar result was obtained independently by [Yaspan 1961], who also derives an expression for $P_1(x, y_1, \ldots, y_L)$ and investigates the sensitivity of y_L with respect to $x, y_1, \ldots, y_{L-1}$. Moreover, he indicates how to modify the myopic policy if there is a positive set-up cost ($K > 0$).

The computation of the stockout probability $P_1(x, y_1, \ldots, y_L)$ is not straightforward and illustrates the difficulties of the lost-sales assumption. Define

$$I_n \quad := \quad \text{on-hand inventory minus shortages at the end of period } n \quad (n = 1, 2, \ldots),$$

then, by definition,

$$P_1(x, y_1, \ldots, y_{L-1}, z) = \Pr\{I_{L+1} < 0\}. \tag{4.8}$$

It is easily seen that the random variable I_{L+1} satisfies the recursive relation

$$\begin{aligned} I_1 &= x - X_1; \\ I_n &= I_{n-1}^+ + y_{n-1} - X_n \quad (n = 2, \ldots, L+1) \end{aligned} \tag{4.9}$$

($x^+ := \max\{x, 0\}$). As pointed out by [Nahmias 1979] (p. 906), (4.9) is known as a Lindley equation in queueing theory; specifically, the process $\{I_n^+\}$ also describes the waiting times of successive customers in a G/G/1 queue with service times $\{x, y_1, \ldots\}$ and interarrival times $\{X_1, X_2, \ldots\}$. Using a result of Lindley (see e.g. [Lindley 1952]),

$$I_{L+1} = \max\{y_L - X_{L+1}, y_{L-1} + y_L - S_{L,L+1}, \ldots, x + \sum_{n=1}^{L} y_n - S_{L+1}\}, \tag{4.10}$$

it follows that

$$\Pr\{I_{L+1} < 0\} = \Pr\{X_{L+1} > y_L, S_{L,L+1} > y_{L-1} + y_L, \ldots, S_{L+1} > x + \sum_{n=1}^{L} y_n\}. \tag{4.11}$$

This result was also proven by [Yaspan 1961] (p. 379), using sample path arguments to show that the two events are identical. In [Yaspan 1972] he elaborates (4.11) for the case of i.i.d. normally distributed demand, showing how the computation of this probability can be reduced to operations on a standard multinormal distribution. A probabilistic and more complicated proof of (4.11) appears in Appendix 5.3 of [Rutten 1995] and in [Regterschot et al. 1992].

In [Morton 1969b] the fixed-stockout-probability (FSP) policy is compared to the optimal policy for $L = 1$ and $L = 2$, and (discretized) normal, exponential and "long-tail" demand distributions, indicating that the FSP policy is near optimal. The FSP policy is generalized to the case of a positive set-up cost ($K > 0$) in [Nahmias 1979], and again compared to the optimal policy for $L = 1$ and $L = 2$, using (discrete) uniform, Poisson and geometric demand distributions. It turns out that the percentage errors are considerably higher for $L = 2$, but decrease with the set-up cost K. The FSP policy was also studied by [Rutten 1995] (section 5.3) and [van Donselaar et al. 1996], who focus on the case of an Erlang demand distribution and a target P_1 service level in every period. They use simulation to compare the FSP policy, which induces a variable reorder level, to a "fixed reorder level" (R, S)-policy. Their conclusion is that, although the FSP policy obviously requires less inventory to obtain the target service level, the differences between the two (heuristic) policies are generally small.

The (R, S)-policy with $R = 1$ (order up to S every period) is discussed by several authors, not only because of its simplicity but also because it is known to be optimal for the periodic-review backorder model with $K = 0$. Karlin and Scarf ([Arrow et al. 1958], 10.4) analyse the case of a fixed lead time and i.i.d. exponentially distributed demand. [Hadley&Whitin 1963] (section 5-13) stress the intrinsic difficulty of the case $L > 1$ and $N > 1$ (more than one order outstanding), and illustrate this by presenting a Markov chain analysis of the $(1, S)$-policy for $0 < L < 1$ and i.i.d. Poisson demand. A similar approach was also used by [Morse 1959] to derive the steady state distribution of on-hand inventory for $L = 1$ and a general discrete demand distribution (both for the backorders and the lost-sales case), and extended by [Gaver 1959] to find the "base-stock" levels that minimize an average cost or discounted cost function. [Pressman 1977] generalizes the Markov chain approach to the case $L > 1$, using the term "order-level-scheduling-period" system and assuming that demand is distributed uniformly within each period (this is important for determining the holding costs). He proves that the number of states of the resulting L-dimensional Markov chain is $\binom{S+L}{L}$. Moreover, he indicates that his approach can be extended to a (n, S)-policy, i.e., order up to S every n periods, by distinguishing between the cases $L \leq n$ and $L > n$ (see [Pressman 1968]).

In the presence of a set-up cost it is usually undesirable to place an order every period, and then a reorder level policy, like (s, Q) or (s, S), is more appropriate. [Wagner 1962] (section 2.3) focuses on (s, S)-policies with at most one outstanding order ($N = 1$), and uses Markov chain theory to obtain the stationary distribution of on-hand inventory for various values of L, s and S. [Hill 1997] computes various performance measures (including expected average costs) for a general state-dependent ordering policy through an embedded Markov chain on order-arrival epochs, also assuming that $N = 1$. [Cohen et al. 1988] consider a more general model with two priority classes of customers and $N \geq 1$, aiming to find the best (s, S)-policy subject to a P_2 service level constraint (in the context of a multi-echelon logistics system). To this end, they develop renewal-theoretic approximations and an algorithm requiring that $S - s$ is considerably larger than the average demand per period. The service-constrained problem (without priority classes) was also studied by [Mitchell 1986], who utilizes a Brownian motion model.

Continuous-review models

The global optimal policy for the continuous-review case is even more complicated than for the periodic-review case, because the number of outstanding orders is not bounded (in the periodic-review case it is bounded by L). Moreover, the optimal policy will not only depend on the size, but - unless the lead time is exponentially distributed - also on the residual lead time of all outstanding orders. Consequently, even for the simplest case of a Poisson demand process and exponential lead times, a complete state description is of infinite dimension. As noted before, these difficulties are overcome by assuming that at most one order may be outstanding at any time ($N = 1$). When considering some fixed policy it is often possible to guarantee this a priori by an appropriate choice of the policy parameters; for an (s, Q)- or (s, S)-policy it suffices to assume that $Q > s$ or $S - s > s$, respectively (see e.g. [Hadley&Whitin 1963], p. 197). Note that for an unconstrained (s,Q)-policy the maximal number of outstanding orders is $\lfloor 1+\frac{s}{Q}\rfloor$. The only case where the assumption $N=\infty$ imposes no difficulties is that of a Poisson demand process and a "one-for-one" $(S-1,S)$-policy (see [Arrow et al. 1958], 16.1; [Hadley&Whitin 1963], 4-13; [Smith 1977]; [Kalpakam&Arivarignan 1989]). This policy, frequently used in spare-parts inventory systems, gives rise to an Erlang loss system and hence only depends on the lead time distribution through the mean (see e.g. [Palm 1938]). In the next two paragraphs we review the literature on continuous-review lost-sales models with at most one order outstanding.

In [Hadley&Whitin 1963] (section 4-11) an (s,Q)-policy with $Q>s$ is analysed for the case of constant lead times and a Poisson demand process. Using the fact that the resulting stochastic process is regenerative on reorder points (since demand is unit-sized), an expression for the expected average costs per unit of time is derived. [Archibald 1981] extends this analysis to compound Poisson demand (with a general discrete compounding distribution) and a fixed (s,S)-policy, by constructing an embedded Markov chain on order-arrival epochs. The optimal (s,S) pairs are computed for various (mostly erratic) compounding distributions and compared to the globally optimal (s,S)-policy from the backorder case, revealing that the penalty of using the backorder solution is generally small. The same model is covered by [Hill 1997], who uses a similar approach and allows for a more general state-dependent ordering policy. [Buchanan&Love 1985] solve the case of a Poisson demand process and Erlang-k distributed lead times under an (s,Q)-policy, using a two-dimensional Markov chain with the inventory level and the number of completed stages of the lead time as state variables. For the special case of exponential lead times ($k=1$) they derive a relatively simple expression for the expected average costs. Since the computations are not limited to small values of k, they can also handle the constant lead time case by letting $k\to\infty$.

Under the assumption $N=1$ and a (compound) Poisson demand process the continuous-review lost-sales model can be formulated as a semi-Markov decision process in order to find an optimal policy. This direction is followed by Johansen and Thorstenson, who assume unit-sized Poisson demand and allow for generic stochastic lead times. They focus on the undiscounted case and Erlang distributed lead times in [Johansen&Thorstenson 1993], while in [Johansen&Thorstenson 1996] they treat the discounted case with constant or exponential lead times. In [Johansen&Thorstenson 1997] they develop a structured policy iteration algorithm and manage to prove their earlier conjecture that the (s,Q)-policy is optimal under the condition that the distribution of lead time demand is log-concave. In the next chapter we will continue along this line and develop an SMDP-framework that includes compound demand and a delay-limit on backorders ($D>0$).

Due to the complicated nature of the problem, almost no research has been reported on continuous-review models where more than one order may be outstanding ($N>1$). In section 16.3 of [Arrow et al. 1958], Scarf analyses a model with exponential lead times and renewal demand under an (s,S)-policy, and obtains an explicit expression for the generating function of the number of outstanding orders (bounded by $\lfloor\frac{S}{S-s}\rfloor$). Hill has studied an (s,Q)-policy with $Q\le s<2Q$, implying that at most two orders will be outstanding at any time, assuming Poisson demand. In [Hill 1992] he presents an exact

RS	review system	CR	continuous review
		PR	periodic review
D	demand	CD	continuous demand
		DD	discrete demand
DP	demand process	BM	Brownian motion
		CPP	compound Poisson process
		Erl	i.i.d. Erlang
		Exp	i.i.d. exponential
		G	i.i.d. general
		N	i.i.d. normal
		P	i.i.d. Poisson
		PP	Poisson process
		RP	renewal process
L	lead time	Erl	Erlang
		Exp	exponential
H	horizon	FH	finite horizon
		IH	infinite horizon
PM	performance measure	AC	expected average costs
		DC	expected discounted costs
		P_1	probability of no stockout
		P_2	fraction of demand met

Table 4.2: Abbrevations used in Table 4.3

Reference	RS	D	DP	L	N	$c(x)$	H	PM	policy
[Arrow et al. 1958], 10.3	PR	CD	G	1	1	cx	IH	DC	characterization
[Arrow et al. 1958], 10.4	PR	CD	Exp	$\in \mathbb{N}$	∞	cx	IH	DC	(R, S)
[Arrow et al. 1958], 16.3	CR	DD	RP	Exp	∞	–	IH	–	(s, S)
[Morse 1959]	PR	DD	G	1	1	$K+cx$	IH	AC	(R, S)
[Gaver 1959]	PR	DD	G	1	1	$K+cx$	IH	AC/DC	(R, S)
[Yaspan 1961]	PR	CD	G	$\in \mathbb{N}$	∞	$K+cx$	FH	DC	myopic
[Wagner 1962], 2.3	PR	DD	G	$\in \mathbb{N}$	1	–	IH	–	(s, S)
[Hadley&Whitin 1963], 4-11	CR	DD	PP	$\in \mathbb{R}^+$	1	$K+cx$	IH	AC	(s, Q)
[Hadley&Whitin 1963], 5-13	PR	DD	P	$\in (0,1)$	1	$K+cx$	IH	AC	(R, S)
[Pressman 1968]	PR	DD	G	$\in \mathbb{N}$	∞				
[Morton 1969a],[Morton 1969b]	PR	CD	G	$\in \mathbb{N}$	∞	cx	IH	DC	myopic
[Yaspan 1972]	PR	CD	N	$\in \mathbb{N}$	∞	–	FH	P_1	-
[Smith 1977]	CR	DD	PP	random	∞	–	IH	AC	$(S\text{-}1, S)$
[Pressman 1977]	PR	DD	G	$\in \mathbb{N}$	∞	K	IH	AC	(nR, S)
[Nahmias 1979]	PR	CD	G	$\in \mathbb{N}$/random	∞	$K+cx$	IH	DC	myopic
[Archibald 1981]	CR	DD	CPP	$\in \mathbb{R}^+$	1	K	IH	AC	(s, S)
[Buchanan&Love 1985]	CR	DD	PP	Erl	1	K	IH	AC	(s, Q)
[Mitchell 1986]	PR	CD	BM	$\in \mathbb{N}$	∞	$K+cx$	IH	AC+P_2	(s, S)
[Cohen et al. 1988]	PR	DD	G	$\in \mathbb{N}$	∞	$K+cx$	IH	AC+P_2	(s, S)
[Hill 1992]	CR	DD	PP	$\in \mathbb{R}^+$	2	–	IH	P_2	(s, Q)
[Johansen&Thorstenson 1993]	CR	DD	PP	Erl	1	$K+cx$	IH	AC	optimal: (s, Q)
[Hill 1994]	CR	DD	PP	Erl	2	–	IH	P_2	(s, Q)
[Rutten 1995], 5.3	PR	CD	Erl	$\in \mathbb{N}$	∞	–	IH	P_1	myopic/(R, S)
[Mohebbi 1996]	CR	DD	CPP	random	$\in \mathbb{N}$				
[Johansen&Thorstenson 1996]	CR	DD	PP	$\in \mathbb{R}^+$/Exp	1	$K+cx$	IH	AC/DC	optimal: (s, Q)
[Hill 1997]	PR	DD	P	$\in \mathbb{N}$	1	K	IH	AC	general
[Hill 1997]	CR	DD	CPP	$\in \mathbb{R}^+$	1	K	IH	AC	general

Table 4.3: Summary of literature on lost-sales inventory models

numerical approach to determine the P_2 service level for constant lead times, and compares the results with three approximative models. In [Hill 1994] he gives a Markov chain analysis for Erlang-k distributed lead times, showing that the number of states equals $Q(K+1) + (R-Q+1)k(k+1)/2$. Finally, we mention the recent work of [Mohebbi 1996], who presents a modeling framework for the analysis of various continuous-review lost-sales models with compound Poisson demand, variable lead times and possibly multiple orders outstanding.

The survey of lost-sales inventory models is summarized in Table 4.3, using the abbrevations in Table 4.2.

4.3.2 Partial backorder inventory models

In the general framework of section 4.2 the case $D > 0$ corresponds to production/inventory models where customer demand can be backordered until D time units after arrival and is lost thereafter. This is a form of partial backlogging, and the literature on partial backlogging models can be divided into three classes:

I Of the demand arriving when on-hand inventory is zero, a fraction β can be backordered and a fraction $1-\beta$ is lost. This is equivalent to saying that a shortage can be backordered with probability β and is lost with probability $1-\beta$. The extreme cases $\beta = 0$ and $\beta = 1$ correspond to a lost-sales and a full backlogging model, respectively.

II Demand can be backordered if the number of backorders does not exceed b, and is lost otherwise. In other words, the number of backorders in an order cycle is bounded by b. The extreme cases $b = 0$ and $b = \infty$ correspond to a lost-sales and a full backlogging model, respectively.

III Demand can be backordered until time D after arrival, and is lost thereafter. This corresponds to the case of impatient customers or a delay-limit on backorders that we focus on. The impatience or delay-limit D may be deterministic or stochastic. The extreme cases $D = 0$ and $D = \infty$ correspond to a lost-sales and full backlogging model, respectively.

For type-I models see e.g. [Nahmias 1979], [Montgomery et al. 1973] and [Kim&Park 1985]; for a type-II model see [Rabinowitz et al. 1995]. The literature on type-III models is very limited. [Posner&Yansouni 1972] study a class of inventory models with an exponentially distributed delay-limit, and provide a detailed analysis for the case of a Poisson demand process and exponential lead times. [Das 1977] considers a continuous-review $(S-1, S)$ inventory model where customers balk if the waiting time exceeds a fixed time-limit, i.e., a constant delay-limit on backorders. However, he assumes that orders are processed one at a time as in a single-server queue with exponential service times.

4.3.3 Perishable inventory models

The problem of inventory control subject to a delay-limit on backorders is similar in nature to the problem of perishable inventory control. In a perishable inventory model ordered items have a finite (deterministic or stochastic) lifetime and perish after this lifetime.

Unless the lifetime distribution is memoryless it is now necessary to keep track of the residual lifetime of items in inventory, similar to the case of delay-limits where we have to keep track of the residual delay-limit of waiting customers. To put it differently, one could speak of impatient customers on one hand and of "impatient" inventory on the other hand. Specifically, the problem of finding an optimal policy for the periodic-review perishable inventory model with a fixed lifetime m, zero lead time and full backlogging can be modelled as a dynamic program with state variable $(x_1, \ldots, x_{m-1})$, where x_i is the number of items that will perish in i periods.

After the perishable inventory problem was introduced by [Van Zyl 1964], a considerable number of papers have appeared on this topic. Most of these papers assume a fixed lifetime for the items, no set-up cost for ordering ($K = 0$) and full backlogging, while all of them assume instantaneous deliveries ($L = 0$). Optimal ordering policies for the case of a two-period lifetime ($m = 2$) are studied by [Van Zyl 1964] and [Nahmias&Pierskalla 1973], both for the finite- and the infinite-horizon problem. The general case of an m-period lifetime was analysed simultaneously and independently by [Fries 1975] and [Nahmias 1975b], using slightly different cost structures. Since the dynamic programming formulation becomes computationally infeasible if m grows and the optimal policy does not have a simple structure, several myopic approximations were tested (see e.g. [Nahmias 1975a], [Nahmias 1976], [Nandakumar&Morton 1993]). A "base-stock" policy, that keeps the total number of items in inventory constant, is analysed by [Cohen 1976] and [Chazan&Gal 1977]. Just as for lost-sales models, [Nahmias 1978] generalizes some of the results to the case of a positive set-up cost ($K > 0$). An exhaustive survey of the literature on perishable inventory problems (until 1982), including models with random lifetimes, can be found in [Nahmias 1982]. Apparently, not much subsequent research has been done since then. Of special interest is the extension to positive order lead times ($L > 0$), but this problem turns out to be extremely difficult.

4.4 Two general results

In standard inventory models with $D = 0$ and lost sales it is impossible to avoid lost sales if demand is not bounded from above; one can set the fraction of demand satisfied from stock on hand arbitrarily close to 1, but never equal to 1. On the contrary, if $D > 0$ it is possible in some cases to avoid lost sales alltogether, e.g., if the production lead time is negligible and $M = \infty$. To illustrate this point, consider the case where $D > 0$ and $L > 0$ are constant, and define P_2 as the expected fraction of demand that is satisfied eventually.

Theorem 4.1 *Let D and L be positive constants, $N \geq 1$ integer and $M = \infty$, and suppose that the total demand in an interval of length T is not bounded from above. Then there exists a policy for which $P_2 = 0$ if and only if*

$$\frac{L}{D} \leq \frac{N}{N+1}. \tag{4.12}$$

Proof. To prove the "only if" part we construct a policy for which $P_2 = 1$ and prove that (4.12) must hold. Clearly, there is no such policy if $L \geq D$, so that we can restrict ourselves to the case $L < D$. Define t_i as the time that the i^{th} production run is started, and suppose w.l.o.g. that $t_1 = 0$. Since the demand in $[0, t_2)$ must be completed before time D, the next run must start before time $D-L$. Hence $t_2 \leq D-L$, and this run includes all demand that has arrived in the interval $[0, t_2)$. Similarly, the demand in $[t_2, t_3)$ must be completed before time t_2+D, so that $t_3 \leq t_2+D-L \leq 2(D-L)$. Continuing in this fashion we find that $t_i \leq t_{i-1}+D-L \leq (i-1)(D-L)$ for $i = 2, 3, \ldots$, with run i including all demand in $[t_{i-1}, t_i)$. Since there are only N machines, the $(N+1)^{\text{th}}$ run cannot start before the first run has finished, or $t_{N+1} \geq L$. Together with $t_{N+1} \leq N(D-L)$ this implies that $L \leq N(D-L)$, or $\frac{L}{D} \leq \frac{N}{N+1}$. For the "if" part, just note that the policy above with $t_i = (i-1)(D-L)$ $(i = 2, 3, \ldots)$ is feasible if (4.12) holds. □

Obviously, even if a policy without lost sales exists, it may still be advantageous (depending on the cost parameters and the state of the system) to incur lost sales occasionally. In other words, the minimum cost policy does not necessarily avoid lost sales. In the extreme case where the penalty costs are very low, the optimal policy will be not to produce at all and let all demand become lost sales (i.e., $P_2 = 0$). On the other hand, if the penalty costs are sufficiently high and it is possible to avoid lost sales (i.e., if $M = \infty$ and $\frac{L}{D} \leq \frac{N}{N+1}$), then the optimal policy will be such that $P_2 = 1$. In this respect it is also interesting to look for the best policy within the class of policies with $P_2 = 1$. The simplest policy within this class is the policy constructed in the proof of Theorem 4.1: start a batch for $\sum_{n=(i-1)(L-D)+1}^{i(L-D)} X_n$ items at the start of period $i(L-D)+1$ $(i = 1, 2, \ldots)$. Under this "just-in-time" (JIT) policy a new batch is started every $L-D$ periods and neither penalty nor holding costs are incurred, so that the average costs are given by

$$g_{\text{JIT}} = \frac{K}{D-L}. \tag{4.13}$$

This policy can be seen as a generalization of the OB-policy for the discrete-time service model with positive batch-service times (see section 2.2).

Multi-machine models may lead to substantial cost reductions compared to single-machine models, but in some cases the assumption $N = 1$ can be made without loss of generality.

Theorem 4.2 *Let D and L be positive constants, $N \geq 1$ integer and $M = \infty$. If $D \geq 2L$, then there is at most one production batch outstanding at any time under an optimal policy.*

Proof. Suppose that at time 0 a new production batch is started that will be completed at time L. Since the size of this batch is not bounded ($M = \infty$), it will at least include all waiting demand. Now the next batch will not be started before time $D-L$, because there are no waiting costs and all demand in $(0, D-L]$ can be satisfied from this batch. It follows that a new batch will never be started before the previous batch is completed if $D-L \geq L$, or $D \geq 2L$. □

Note that the condition $M = \infty$ is crucial, because if $M < \infty$ then a new batch will be started earlier when the number of waiting demands has reached the level M before time $D-L$. As a direct consequence of Theorem 4.2, the state space can be reduced considerably for a periodic-review model with $M = \infty$ and $D \geq 2L$; we will come back to this in Chapter 6.

4.5 Outline of the remainder of the thesis

In the remainder of this thesis we will consider the following models within the general framework of section 4.2 (see Table 4.2 for the abbreviations):

(a) $D = 0$, $L > 0$, $N = 1$, $M = \infty$, PR (Chapter 5, model P);

(b) $D = 0$, $L > 0$, $N = 1$, $M = \infty$, CR, demand CPP (Chapter 5, model C);

(c) $0 < D \leq L$, $N = 1$, $M = \infty$, PR (Chapter 5, model PB);

(d) $0 < D \leq L$, $L \geq 1$, $N = 1$, $M = \infty$, CR, demand CPP (Chapter 5, model CB);

(e) $D > L \geq 0$, $N = \infty$, $M = \infty$, PR (Chapter 6, model PU);

(f) $D > L \geq 0$, $N = 1$, $M = \infty$, PR (Chapter 6, model PC);

(g) $D \geq 0$, $L \geq 0$, $N \leq \infty$, $M \leq \infty$, PR (Chapter 7).

As argued in section 4.1, a dichotomy arises depending on whether $D \leq L$ or $D > L$: if $D \leq L$ then production to order is precluded, if $D > L$ then production to order is possible and a queue of waiting customers may build up. In Chapter 5 we consider the case $0 \leq D \leq L$, and focus on single-machine production/inventory models with ample production capacity, or, equivalently, inventory models with at most one outstanding order ($N = 1$, $M = \infty$). We formulate a general SMDP that captures all of the models (a)–(d). The assumption $N = 1$ guarantees that we can use a one-dimensional state space (on-hand inventory). We also investigate some heuristic policies, including the well-known (s, Q)-policy. In Chapter 6 we turn to the case $D > L$, that is closely related to the service model of Part I. The state space for model (e) must also include the number of demands with a residual delay-limit of i periods ($i = 0, \ldots, D-L-1$), and the problem is when to start a new production run and how many items to produce. For the timing of the production run we can use similar policies as in Chapter 2, while one additional parameter is needed that determines the number of items to produce in excess of waiting demand. We deal with the case $N = \infty$ as well as the case $N = 1$; although the state space is the same for both cases, the model formulation for $N = \infty$ is considerably easier (in contrast to the case $D \leq L$ in Chapter 5).

In Chapter 7 we relax the assumptions of Chapters 5 and 6, and present a general periodic-review model for the multi-machine capacitated problem ($N \geq 1$, $M \leq \infty$). We formulate a MDP with a $(\max\{D, 1\} + \max\{L, 1\} - 1)$-dimensional state space; not surprisingly, the optimal policy may be extremely complex and is only within reach for small values of $L+D$. We illustrate this by providing some illustrative numerical examples for combinations of D and L with $L + D \leq 1$. We also discuss an appealing periodic (static) policy, that does not use detailed information on the state of the system and is especially useful when other (dynamic) policies are computationally infeasible: start a new production run for Q items every T periods.

The analysis of various models within the general framework presented here is by no means complete; as a matter of fact, it is only a first step towards a more thorough treatment. A lot of interesting models and policies remain to be investigated, especially for the capacitated problem. Therefore we conclude this thesis in Chapter 8 with possible directions for further research, as well as some general conclusions.

4.6 Preliminaries: Discrete renewal theory

In this section we present some results from discrete renewal theory that we will use in Part II of this thesis. Renewal processes arise naturally when dealing with periodic-review inventory models with an i.i.d. demand process, where the interrenewal times correspond to successive demands (see e.g. [Tijms 1994], Example 1.1.2). Since we focus on discrete-demand models throughout this thesis, we need discrete renewal processes. Discrete renewal theory (or recurrent event theory) is fairly well studied in the literature; see e.g. [Feller 1968] (Chapter XIII), [Hunter 1983] (Chapter 3) and [Port 1994] (Chapter 33). However, they all restrict attention to the case where the interrenewal time distribution $\{q_k\}$ has no probability mass in zero, which is inappropriate for a demand distribution. Although most of the results for the case $q_0 = 0$ carry over to the case $q_0 > 0$ by using the transformation $q'_k := \frac{q_k}{1-q_0}$ $(k = 1, 2, \ldots)$ (see also [Feller 1968], section XIII.10, footnote 8), there are some exceptions. In the following we restate the most important results for the case $q_0 = 0$, and then apply the results to the special cases of geometric and Poisson interrenewal times.

4.6.1 General results

Consider a discrete renewal process generated by a sequence of i.i.d. discrete random variables $\{X_n;\ n = 1, 2, \ldots\}$, with

$$q_k := \Pr\{X_n = k\}, \quad Q_k := \sum_{j=0}^{k} q_j, \quad \bar{Q}_k := \sum_{j=k}^{\infty} q_j \quad (k = 0, 1, \ldots).$$

It is convenient to interpret X_n as the number of periods between the $(n-1)^{\text{th}}$ and n^{th} renewal (interrenewal time). Define the partial sums $S_n := \sum_{m=1}^{n} X_m$ (the index of the period in which the n^{th} renewal occurs), with

$$q_k^{(n)} := \Pr\{S_n = k\}, \quad Q_k^{(n)} := \sum_{j=0}^{k} q_j^{(n)} \quad \bar{Q}_k^{(n)} := \sum_{j=k}^{\infty} q_j^{(n)} \quad (k = 0, 1, \ldots).$$

Now the total number of renewals in periods $1, \ldots, i$ and the discrete renewal function are given by

$$N_i := \max\{n : S_n \le i\} \quad (i = 0, 1, \ldots); \tag{4.14}$$
$$M_i := E\{N_i\} \quad (i = 0, 1, \ldots), \tag{4.15}$$

respectively. Note that N_0 is geometrically distributed with parameter $1 - q_0$, so that $M_0 = \frac{q_0}{1-q_0}$.

Conditioning on X_1 gives a discrete renewal equation for M_i,

$$M_i = Q_i + \sum_{k=0}^{i} q_k M_{i-k} \quad (i = 0, 1, \ldots). \tag{4.16}$$

Since

$$N_i \geq n \iff S_n \leq i \quad (i = 0, 1, \ldots;\ n = 0, 1, \ldots), \tag{4.17}$$

we have that

$$M_i = \sum_{n=1}^{\infty} \Pr\{N_i \geq n\} = \sum_{n=1}^{\infty} \Pr\{S_n \leq i\} = \sum_{n=1}^{\infty} Q_i^{(n)}. \tag{4.18}$$

The following theorem gives the solution of the general discrete renewal equation, of which (4.16) is a special case.

Theorem 4.3 *Let $\{a_i;\ i = 0, 1, \ldots\}$ be a sequence of non-negative numbers. Then the discrete renewal equation*

$$b_i = a_i + \sum_{k=0}^{i} q_k b_{i-k} \quad (i = 0, 1, \ldots), \tag{4.19}$$

has a unique solution that can be written as

(i) $b_i = \dfrac{a_i}{1 - q_0} + \sum_{k=1}^{i} a_{i-k} \sum_{n=1}^{\infty} q_k^{(n)} \quad (i = 0, 1, \ldots);$

(ii) $b_i = \dfrac{a_i}{1 - q_0} + \sum_{k=1}^{i} a_{i-k}(M_k - M_{k-1}) \quad (i = 0, 1, \ldots);$

(iii) $b_i = a_i + \sum_{n=1}^{\infty} E\{a_{i-S_n}^{+}\} \quad (i = 0, 1, \ldots;\ a_i := 0 \text{ if } i < 0).$

Proof. We first prove by induction that

$$b_i = a_i + \sum_{k=0}^{i} a_{i-k} \sum_{n=1}^{\infty} q_k^{(n)} \quad (i = 0, 1, \ldots). \tag{4.20}$$

It immediately follows from (4.19) that $b_0 = \frac{a_0}{1-q_0}$, while substituting $i = 0$ in (4.20) also gives

$$b_0 = a_0 + a_0 \sum_{n=1}^{\infty} q_0^n = \frac{a_0}{1 - q_0}.$$

Next suppose that (4.20) holds for $j = 0, \ldots, i - 1$. Then it follows from (4.19) that

$$\begin{aligned}
b_i &= a_i + q_0 b_i + \sum_{k=1}^{i} q_k \left(a_{i-k} + \sum_{l=0}^{i-k} a_{i-k-l} \sum_{n=1}^{\infty} q_l^{(n)} \right) \\
&= a_i + q_0 b_i + \sum_{k=1}^{i} a_{i-k} q_k + \sum_{n=1}^{\infty} \sum_{m=1}^{i} a_{i-m} \sum_{k=1}^{m} q_k q_{m-k}^{(n)} \\
&= a_i + q_0 b_i + \sum_{k=1}^{i} a_{i-k} q_k + \sum_{m=1}^{i} a_{i-m} \sum_{n=1}^{\infty} \left(q_m^{(n+1)} - q_0 q_m^{(n)} \right) \\
&= q_0 b_i + (1 - q_0) \left(a_i + \sum_{k=1}^{i} a_{i-k} \sum_{n=1}^{\infty} q_k^{(n)} \right),
\end{aligned}$$

and solving for b_i yields (4.20). Now (i) follows from (4.20) by evaluating the term for $k = 0$ and the fact that $q_0^{(n)} = q_0^n$, (ii) follows from (i) and (4.18), and (iii) follows from (4.20) by noting that $\sum_{k=0}^{i} a_{i-k} q_k^{(n)} = E\{a_{i-S_n}^+\}$ if we set $a_i := 0$ for $i < 0$. □

The well-known Discrete Renewal Theorem ([Feller 1968], section XIII.10, Theorem 1; [Hunter 1983], Theorem 3.3.6; [Port 1994], Proposition 33.2) is easily generalized to the case $q_0 > 0$.

Theorem 4.4 *(Discrete Renewal Theorem)*
If $\{b_i;\ i = 0, 1, \ldots\}$ satisfies the discrete renewal equation (4.19), then

$$\lim_{i\to\infty} b_i = \frac{\sum\limits_{j=0}^{\infty} a_j}{\mu}.$$

Proof. Using the transformation $q_i' := \frac{q_i}{1-q_0}$ $(i = 1, 2, \ldots)$ we can rewrite (4.19) as

$$b_i = \frac{a_i}{1-q_0} + \sum_{k=1}^{i} q_k' b_{i-k} \quad (i = 0, 1, \ldots),$$

and apply Theorem 1 in section XIII.10 of [Feller 1968] to obtain

$$\lim_{i\to\infty} b_i = \frac{\sum\limits_{j=0}^{\infty} \dfrac{a_j}{1-q_0}}{\sum\limits_{j=1}^{\infty} j q_j'} = \frac{\sum\limits_{j=0}^{\infty} a_j}{\mu}. \quad \square$$

Next we turn to the forward recurrence time in the n^{th} period,

$$\gamma_i := S_{N_i+1} - i \quad (i = 0, 1, \ldots), \tag{4.21}$$

with pmf $g_k^{(i)} := \Pr\{\gamma_i = k\}$ ($k = 1, 2, \ldots$; note that $g_0^{(i)} = 0$ by definition). Moreover, the asymptotic forward recurrence time γ_∞ is defined through

$$g_k := \lim_{i\to\infty} g_k^{(i)} = \Pr\{\gamma_\infty = k\} \quad (k = 1, 2, \ldots). \tag{4.22}$$

Theorem 4.5 *(i)* $g_k^{(i)} = \dfrac{q_{i+k}}{1-q_0} + \sum\limits_{j=1}^{i} q_{i+k-j} \sum\limits_{n=1}^{\infty} q_j^{(n)}$ $(i \geq 0;\ k = 1, 2, \ldots)$;

(ii) $g_k = \dfrac{\bar{Q}_k}{\mu}$ $(k = 1, 2, \ldots)$.

Proof. Conditioning on X_1 yields a discrete renewal equation for $g_k^{(i)}$,

$$g_k^{(i)} = \sum_{j=0}^{i} q_j g_k^{(i-j)} + q_{i+k} \quad (i = 0, 1, \ldots;\ k \geq 1). \tag{4.23}$$

Setting $a_i := q_{i+k}$ $(i \geq 0)$, (i) follows from Theorem 4.3(i) and (ii) from Theorem 4.4. □

4.6.2 Some special cases

Bernoulli trials

The simplest example of a discrete renewal process is a sequence of Bernoulli trials with a success probability of r at any trial, where a renewal corresponds to a success. In this case $X_n - 1 \sim \mathrm{G}(r)$ $(n = 1, 2, \ldots)$ and N_i is just the number of successes in the first i trials; the following results are well known.

Theorem 4.6 *(i)* $N_i \sim B(i, r)$ *(*$i = 1, 2, \ldots$*), i.e.,*

$$\Pr\{N_i = n\} = \binom{i}{n} r^n (1-r)^{i-n} \quad (i \geq 1;\ n = 0, \ldots, i);$$

(ii) $M_i = ri$ *(*$i = 1, 2, \ldots$*).*

Proof. See e.g. [Hunter 1983], Example 3.4.1.

Geometric interrenewal times

Next we consider the case where $X_n \sim \mathrm{G}(r)$ $(n = 1, 2, \ldots)$, i.e.,

$$q_k = (1-r)^k r, \quad Q_k = 1 - (1-r)^{k+1}, \quad \bar{Q}_k = (1-r)^k \quad (k = 0, 1, \ldots) \tag{4.24}$$

(note the positive probability mass in 0). Taking the n-fold convolution of the geometric distribution gives the negative binomial distribution, whence $S_n \sim \mathrm{NB}(n, r)$ and

$$q_k^{(n)} = \Pr\{S_n = k\} = \binom{n+k-1}{n} r^n (1-r)^k \quad (k = 0, 1, \ldots). \tag{4.25}$$

Theorem 4.7 *(i)* $N_i \sim NB(i+1, 1-r)$ *(*$i = 0, 1, \ldots$*), i.e.,*

$$\Pr\{N_i = n\} = \binom{i+n}{n} r^n (1-r)^{i+1} \quad (i = 0, 1, \ldots;\ n = 0, 1, \ldots);$$

(ii) $M_i = \dfrac{1-r}{r}(i+1)$ *(*$i = 0, 1, \ldots$*);*
(iii) $\gamma_i - 1 \sim G(r)$ *(*$i = 0, 1, \ldots$*), i.e.,*

$$g_k^{(i)} = (1-r)^{k-1} r \quad (i = 0, 1, \ldots;\ k = 1, 2, \ldots).$$

Proof. (i) Using $\binom{n+k-1}{k} = \binom{n+k}{k} - \binom{n+k-1}{k-1}$ $(k = 1, 2, \ldots)$, we find that

$$\begin{aligned}
\Pr\{N_i = n\} &= \Pr\{S_n \leq i\} - \Pr\{S_{n+1} \leq i\} \\
&= \sum_{k=0}^{i} \binom{n+k-1}{k} r^n (1-r)^k - \sum_{k=0}^{i} \binom{n+k}{k} r^{n+1} (1-r)^k \\
&= \sum_{k=0}^{i} \binom{n+k}{k} r^n (1-r)^{k+1} - \sum_{k=1}^{i} \binom{n+k-1}{k-1} r^n (1-r)^k \\
&= \binom{n+i}{i} r^n (1-r)^{i+1} \quad (i = 0, 1, \ldots;\ n = 0, 1, \ldots),
\end{aligned}$$

in which we recognize the pmf of the NB$(i+1, 1-r)$ distribution.

The result also follows from Theorem 4.6(i) by noting that

$$\Pr\{N_i = n\} = \Pr\{\tilde{N}_{i+n} = n\} = \binom{i+n}{n} r^n (1-r)^i \quad (i = 0, 1, \ldots;\ n = 0, 1, \ldots),$$

where $\tilde{N}_i$ denotes the number of successes in i Bernoulli trials.

(ii) This follows directly from (i). Alternatively, we can use (4.18), (4.25) and the identity

$$\sum_{n=1}^{\infty} \binom{n+k-1}{k} r^n = \frac{r}{(1-r)^{k+1}} \quad (k = 0, 1, \ldots) \tag{4.26}$$

to obtain

$$M_i = \sum_{n=1}^{\infty} Q_i^{(n)} = \sum_{k=0}^{i} (1-r)^k \sum_{n=1}^{\infty} \binom{n+k-1}{n} r^n = \frac{1-r}{r}(i+1) \quad (i \geq 0).$$

(iii) Applying Theorem 4.5(i) and using (4.26) gives

$$\begin{aligned}
g_k^{(i)} &= (1-r)^{i+k-1} r + \sum_{j=1}^{i} (1-r)^{i+k-j} r \sum_{n=1}^{\infty} \binom{n+j-1}{n} r^n (1-r)^j \\
&= (1-r)^{i+k-1} r + \sum_{j=1}^{i} (1-r)^{i+k-j-1} r^2 \\
&= (1-r)^{i+k-1} r + (1-r)^{k-1} r \left(1 - (1-r)^i\right) \\
&= (1-r)^{k-1} r \quad (i = 0, 1, \ldots;\ k = 1, 2, \ldots). \quad \square
\end{aligned}$$

Theorem 4.7 can be explained probabilistically by thinking in terms of successes and failures. Consider a sequence of Bernoulli trials with success probability r, and let S_n denote the number of failures preceeding the n^{th} success. Then it is not difficult to see that $N_i = \max\{n : S_n \leq i\}$ is just the number of successes preceeding the $(i+1)^{\text{th}}$ failure, which is NB$(i+1, 1-r)$ distributed.

Interestingly, a discrete renewal process with G(r) distributed interrenewal times is equivalent to a compound renewal process with unit interrenewal times and a G$(1-r)$ compounding distribution. This useful result is stated formally in the following theorem.

Theorem 4.8 *N_i can be decomposed into*

$$N_i = \sum_{j=0}^{i} Y_j \quad (i = 0, 1, \ldots),$$

with $Y_j \sim G(1-r)$ $(j = 0, \ldots, i)$ and $Y_0, \ldots, Y_i$ mutually independent.

Proof. See Appendix 4.A.

Poisson interrenewal times

Finally, we consider the case where $X_n \sim \mathrm{P}(\lambda)$ $(n = 1, 2, \ldots)$.

Theorem 4.9 *(i)* $N_i = \lceil B_{i+1} \rceil - 1$, *with* $B_i \sim$ *Erlang*(i, λ) $(i = 0, 1, \ldots)$*;*
(ii)

$$\begin{aligned} \Pr\{N_i = n\} &= \int_n^{n+1} \frac{\lambda^{i+1} x^i e^{-\lambda x}}{i!}\, dx \\ &= \sum_{k=0}^{i} e^{-n\lambda} \frac{\lambda^k}{k!} \left(n^k - e^{-\lambda}(n+1)^k\right) \quad (i = 0, 1, \ldots;\ n = 0, 1, \ldots); \end{aligned}$$

(iii)

$$\begin{aligned} M_i &= \sum_{k=0}^{i} \sum_{n=0}^{k} \sigma_k^{(n)} \frac{n!}{k!} \frac{\lambda^k e^{-n\lambda}}{(1 - e^{-\lambda})^{n+1}} - 1 \\ &= \sum_{k=0}^{i} \sum_{n=0}^{k} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{(j\lambda)^k}{k!} \frac{e^{-n\lambda}}{(1 - e^{-\lambda})^{n+1}} - 1 \quad (i = 0, 1, \ldots). \end{aligned}$$

Proof. (i) Consider a Poisson process $\{N(t), t \geq 0\}$, i.e., $N(t) = \max\{n : B_n \leq t\}$ with $B_n = \sum_{i=1}^n A_i$ and $A_i \sim \mathrm{Exp}(\lambda)$. If we set $X_n := N(n) - N(n-1)$ $(n = 1, 2, \ldots)$ then $\{X_n;\ n = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables with a $\mathrm{P}(\lambda)$ distribution, and hence $\{X_n\}$ generates a discrete renewal process with Poisson interrenewal times. Now (i) follows by observing that

$$\{N_i \geq n\} \iff \{S_n \leq i\} \iff \{N(n) \leq i\} \iff \{B_{i+1} > n\}. \tag{4.27}$$

(ii) The first equality follows directly from (4.27), while the second equality follows from (4.17) and the fact that $S_n \sim \mathrm{P}(n\lambda)$.
(iii) Using (4.18) we have that

$$M_i = \sum_{n=1}^{\infty} Q_i^{(n)} = \sum_{n=1}^{\infty} \sum_{k=0}^{i} e^{-n\lambda} \frac{(n\lambda)^k}{k!} = \sum_{k=0}^{i} \frac{\lambda^k}{k!} \sum_{n=1}^{\infty} n^k (e^{-\lambda})^n. \tag{4.28}$$

Now we can use the identity

$$n^k = \sum_{i=0}^{\min(n,k)} \sigma_k^{(i)} \frac{n!}{(n-i)!}, \tag{4.29}$$

where

$$\sigma_k^{(i)} := \frac{1}{i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^k \quad (k = 1, 2, \ldots;\ i = 1, \ldots, k) \tag{4.30}$$

are the *Stirling numbers of the second kind* (see e.g. [Gradshteyn and Ryzhik 1994], section 9.74). Next, using (4.29), (4.30) and the fact that

$$\sum_{n=i}^{\infty} \frac{n!}{(n-i)!} a^{n-i} = \frac{d^i}{da^i} \sum_{n=0}^{\infty} a^n = \frac{d^i}{da^i} \frac{1}{1-a} = \frac{i!}{(1-a)^{i+1}}, \tag{4.31}$$

we obtain

$$\sum_{n=0}^{\infty} n^k a^n = \sum_{n=0}^{\infty} \sum_{i=0}^{\min\{n,k\}} \sigma_k^{(i)} \frac{n!}{(n-i)!} a^n = \sum_{i=0}^{k} \sigma_k^{(i)} a^i \sum_{n=i}^{\infty} \frac{n!}{(n-i)!} a^{n-i} = \sum_{i=0}^{k} \sigma_k^{(i)} \frac{i! a^i}{(1-a)^{i+1}}. \quad (4.32)$$

Finally, substituting (4.32) with $a = e^{-\lambda}$ into (4.28) gives the first equality, and using (4.30) the second. □

Appendix 4.A: Proof of Theorem 4.8

Define $Y_0 := N_0$ and $Y_j := N_j - N_{j-1}$ ($j = 1, 2, \ldots$), so that Y_j is the number of renewals at time j. Now it immediately follows from Theorem 4.7 that $Y_j \sim \mathrm{G}(1-r)$ for any j, and it remains to prove that $Y_0, \ldots, Y_i$ are mutually independent. To this end we will prove that, for all $j = 1, \ldots, i$,

$$\Pr\{Y_j = k_j \mid Y_0 = k_0, \ldots, Y_{j-1} = k_{j-1}\} = \Pr\{Y_j = k_j\} \quad (k_0, \ldots, k_j \geq 0). \tag{4.33}$$

Let $t := \max\{u \in \{0, \ldots, n-1\} : k_u > 0\}$ and $t := -\infty$ if $k_u = 0$ for all $u = 0, \ldots, n-1$. We need to distinguish between $k_n > 0$ and $k_n = 0$, as well as between $t \geq 0$ and $t = -\infty$. Defining $k := k_0 + \ldots + k_t$, we have for $t \geq 0$ that

$$\begin{aligned}
& \Pr\{Y_j = k_j \mid Y_0 = k_0, \ldots, Y_t = k_t > 0, Y_{t+1} = \cdots = Y_{j-1} = 0\} \\
= \; & \Pr\{X_k = j - t, X_{k+1} = \cdots = X_{k+k_j} = 0, X_{k+k_j+1} > 0 \mid X_k \geq j - t\} \\
= \; & \frac{\Pr\{X_{k+1} = j - t, X_{k+2} = \cdots = X_{k+k_j} = 0, X_{k+k_j+1} > 0}{\Pr\{X_k \geq j - t\}} \\
= \; & \frac{(1-r)^{j-t} r \cdot r^{k_j - 1} \cdot (1-r)}{(1-r)^{j-t}} = r^{k_j}(1-r) \quad (j = 1, \ldots, i;\; k_j \geq 1)
\end{aligned}$$

and

$$\begin{aligned}
& \Pr\{Y_j = 0 \mid Y_0 = k_0, \ldots, Y_t = k_t > 0, Y_{t+1} = \cdots = Y_{j-1} = 0\} \\
= \; & \Pr\{X_k > j - t \mid X_k \geq j - t\} \\
= \; & 1 - r \quad (j = 1, \ldots, i).
\end{aligned}$$

If $t = -\infty$ then

$$\begin{aligned}
& \Pr\{Y_j = k_j \mid Y_0 = \cdots = Y_{j-1} = 0\} \\
= \; & \Pr\{X_1 = j, X_2 = \cdots = X_{k_j} = 0 \mid X_1 \geq j\} \\
= \; & r^{k_j}(1-r) \quad (j = 1, \ldots, i;\; k_j \geq 1)
\end{aligned}$$

and

$$\Pr\{Y_j = 0 \mid Y_0 = \cdots = Y_{j-1} = 0\} = \Pr\{X_1 > j \mid X_1 \geq j\} = 1 - r \quad (j = 1, \ldots, i).$$

Consequently, (4.33) holds for all $j = 1, \ldots, i$ and $k_0, \ldots, k_j \geq 0$. □

Chapter 5

Models precluding production to order ($D \leq L$)

5.1 Introduction

In this chapter we focus on the case where D and L are both constant with $D \leq L$, and we limit ourselves to the class of single-machine models ($N = 1$), i.e., a new production run cannot be started before the previous one is completed. As argued in the previous chapter, the assumption $D \leq L$ precludes production to order: if a production run is started after a customer arrives, then by the time the run is completed the delay-limit of the customer has expired and the demand is lost. Consequently, customer demand can only be satisfied if either stock on hand is available or a current production run is completed within time D.

This setting thus boils down to a production/inventory model where in principal demand is satisfied from stock on hand, with the objective of minimizing the sum of production, holding and penalty costs. As such, the model can also be described as an inventory model with at most one outstanding order and a delay-limit on backorders, i.e., demand can be backordered until the delay-limit is reached and is lost thereafter. This is a form of "partial backordering" that is less commonly studied; see section 4.3.2 for two other types of partial backorder models. Notably, the term "lost sales" should be interpreted in a broad sense and also includes cases where demand can be contracted out or satisfied by other means, against a fixed penalty cost per item. If the penalty cost is relatively low compared to the production costs, then it may even be better not to produce at all and to let all demand "get lost". This phenomenon may occur in any (production/)inventory model with lost sales or partial backordering, since then the expected costs per unit of time associated with a no-production (or no-order) policy are always finite. Obviously, this is not an option for models with complete backordering of demand since in that case, by definition, all demand must be satisfied eventually.

The decision problem now is: when to start a new production run and how much to produce, given the level of stock on hand. Clearly, when stock on hand has dropped to zero while no production run is currently underway, a new production run will be started immediately (provided that the no-production is not optimal, and that no information is available on the time of the next demand arrival). The most natural structure of an optimal policy is that, if stock on hand drops below a certain "reorder" level, a new production run is started at the first opportunity (immediately if no run is underway and immediately after completion of the current run otherwise). Possible production policies include (s, Q) (produce Q items at the first opportunity after stock on hand has dropped to or below s) and (s, S) (produce the quantity that is necessary to increase the inventory position to S at the first opportunity after stock on hand has dropped to or below s). If demand is unit-sized then there is no undershoot of the reorder level, in which case the (s, Q)- and (s, S)-policy are equivalent (with $S = s + Q$).

In section 5.2 we will construct a general semi-Markov decision process (SMDP) through which the optimal production policy can be found. It turns out that for batch demand neither an (s, Q)-policy nor an (s, S)-policy is optimal, but that the production quantity is a complex function of the undershoot. We will assume throughout that both the production capacity and the inventory capacity are sufficiently large to carry out the policy under consideration (note that the maximal stock level is equal to the reorder level plus the maximal production quantity, provided that we confine ourselves to policies with reorder levels).

We consider four different models in this chapter, which all fit into the general SMDP:

P: Periodic review, lost sales ($D = 0$);

C: Continous review, compound Poisson arrivals, lost sales ($D = 0$);

PB: Periodic review, delay-limit on backorders ($D > 0$);

CB: Continous review, compound Poisson arrivals, delay-limit on backorders ($D > 0$).

In the periodic-review models P and PB, X_n represents the total demand in period n and $S_n := \sum_{m=1}^{n} X_m$ the total demand over the first n periods. Demand can either arrive in one batch per period or in a number of batches during the period; specific demand arrival epochs within the period are irrelevant for the state transitions. They do, however, influence the holding costs, and we will assume for the sake of simplicity that all demand arrives at the beginning of a period (as we did in Chapter 2).

The continuous-review models C and CB assume "genuine" batch arrivals, with X_i the size of the i^{th} batch. Denote the times between successive batch arrivals by A_i and let A_i ($i = 1, 2, \ldots$) be exponentially distributed with parameter λ. Define $B_i := \sum_{j=1}^{i} A_j$ (the arrival epoch of the i^{th} batch) and let $N(t)$ be the renewal function generated by $\{A_i\}$, so that $\{N(t)\}$ is a Poisson process with parameter λ. Next define

$$Y(t) := \sum_{i=1}^{N(t)} X_i \quad (t \geq 0), \tag{5.1}$$

then $\{Y(t)\}$ is a compound Poisson process and $Y(t)$ denotes the total demand during the interval $[0, t]$ (which is equivalent to the total demand in an arbitrary interval of length t by the lack of memory of the Poisson process). Let $q_k(t) := \Pr\{Y(t) = k\}$, $Q_k(t) := \sum_{j=0}^{k} q_j(t)$ and $\bar{Q}_k(t) := \sum_{j=k}^{\infty} q_j(t)$ ($k = 0, 1, \ldots$) denote the pmf, the cdf and the tail probabilities of

$Y(t)$, respectively (not to be confused with $q_k^{(n)}$, $Q_k^{(n)}$ and $\bar{Q}_k^{(n)}$ denoting similar quantities for the n-fold convolution of the distribution of X_1). Obviously, $\{Y(t)\}$ reduces to a simple Poisson demand process by setting $q_1 := 1$, in which case $Y(t) = N(t)$ and $q_k(t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$. Another interesting special case is that where X_i has a Poisson distribution; then $Y(t)$ represents a Poisson number of Poisson random variables, for which the distribution is known as the Neyman type A distribution (see e.g. [Johnson&Kotz 1969], section 9.6). We discuss the Neyman type A distribution in detail in Appendix 5.A.

The rest of this chapter is organized as follows. In section 5.2 we describe the aforementioned general SMDP. We then elaborate this SMDP for the lost-sales case ($D = 0$) in section 5.3, where in subsection 5.3.1 we focus on the periodic-review case (model P) and in subsection 5.3.2 on the compound-Poisson case (model C). Next, in section 5.4, we treat the more complicated case of a positive delay-limit on backorders ($0 < D \leq L$); the periodic-review case (model PB) in subsection 5.4.1, the compound-Poisson case (model CB) in subsection 5.4.2. In section 5.5 we illustrate the complexity of the optimal policy by presenting some numerical examples for model P, where it turns out that the optimal production quantity is not even a non-increasing function of on-hand inventory. Because of the lack of structure of the optimal policy, the analysis of intuitive and nicely-structured policies that are close to optimal is important. In section 5.6 we describe some general techniques that can be used to analyse any fixed policy, such as value-determination and an embedded Markov chain approach. In section 5.7 we consider the class of (s, Q)-policies, and we apply these techniques to compute the expected average costs and the stationary distribution of on-hand inventory for a given (s, Q)-policy. We focus on model P, but the other models require only minor modifications. We then introduce the class of so-called (s, S, Q)-policies in section 5.8, a generalization of both the (s, Q)- and the (s, S)-policy. As argued earlier, the no-production policy will be optimal if the penalty cost is small enough, and in section 5.9 we investigate how to compute this critical value for the penalty cost. Finally, in section 5.10, we make a side-step to a continuous-review lost-sales model with unit-sized renewal demand, thereby generalizing some results in [Hadley&Whitin 1963] and [Johansen&Thorstenson 1993]. We conclude the chapter with extensive numerical results for the various models and the various policies in section 5.11.

5.2 A general semi-Markov decision process

In this section we construct a general semi-Markov decision process for all four models introduced in the previous section. We first describe the SMDP for the periodic-review case (models P and PB), and then deal with the modifications needed to handle the compound-Poisson case (models C and CB).

For the periodic-review case, the decision epochs are

- endpoints of review periods in which no production run was underway;
- endpoints of review periods in which a production run was completed.

Clearly, the state space is given by

$$\Omega := \{i \mid i = 0, 1, \ldots\}, \tag{5.2}$$

where i denotes the on-hand inventory at a decision epoch. Moreover, the action space in any state i is given by

$$A(i) := \{a \mid a = 0, 1, \ldots\} \quad (i \in \Omega), \tag{5.3}$$

where action $a = 0$ corresponds to "do not start a production run" and action a to "start a production run for a items" ($a = 1, 2, \ldots$). When a production batch is started ($a > 0$), the next decision epoch is exactly L periods later upon completion of this batch, since no new run can be started in the meantime.

Remark. It is also possible to formulate a MDP with decision epochs at the endpoints of *every* period, by using a two-dimensional state space $\{(i,t) \mid i = 0, 1, \ldots;\ t = 1, \ldots, L\}$ and setting $A(i,t) = \{0\}$ for $t = 1, \ldots, L-1$; see method (v) in section 5.7.

Next we specify, for every state i with corresponding action a, the transition probabilities $p_{ij}(a)$, the transition times $\tau_i(a)$ and the transition costs $c_i(a)$. If no production run is started at the end of a period ($a = 0$), then the next decision epoch is at the end of the following period and we have that

$$\begin{aligned}
p_{ij}(0) &= \Pr\{(i - X_1)^+ = j\} = \begin{cases} \bar{Q}_i & \text{if } j = 0 \\ q_{i-j} & \text{if } 0 < j \le i \\ 0 & \text{else} \end{cases} \quad (i \in \Omega); &(5.4)\\
\tau_i(0) &= 1 \quad (i \in \Omega); &(5.5)\\
c_i(0) &= l(i) \quad (i \in \Omega). &(5.6)
\end{aligned}$$

Here $l(i)$ is the discrete one-period loss function,

$$l(i) := E\big\{h(i - X_1)^+ + p(X_1 - i)^+\big\} = (h + p)\sum_{k=0}^{i-1} Q_k + p(\mu - i) \quad (i = 0, 1, \ldots), \tag{5.7}$$

that can be computed recursively from

$$\begin{aligned}
l(0) &= p\mu; \\
l(i) &= l(i-1) + (h + p)Q_{i-1} - p \quad (i = 1, 2, \ldots). &(5.8)
\end{aligned}$$

For the compound-Poisson case, the decision epochs are

- demand arrival epochs when no production run is underway;
- epochs at which a production run is completed.

It is easily seen that the transition probabilities for $a = 0$ are exactly the same as for the periodic-review case and given by (5.4). The transition times and the transition costs for $a = 0$ only have to be adjusted for the average time between demand arrivals, i.e.,

$$\begin{aligned}
\tau_i(0) &= \frac{1}{\lambda} \quad (i \in \Omega); &(5.9)\\
c_i(0) &= \frac{l(i)}{\lambda} \quad (i \in \Omega), &(5.10)
\end{aligned}$$

with $l(i)$ given by (5.7).

Next suppose that a production run is started at some decision epoch ($a > 0$). Then, by definition, we have that

$$\tau_i(a) = L \quad (i \in \Omega;\ a > 0), \tag{5.11}$$

which means L periods for the periodic-review case and L time units for the continuous-review case. What distinguishes the SMDP-description for the different models is the specification of the transition probabilities and costs for $a > 0$. In sections 5.3 and 5.4 we will complete the description of the SMDP for models P, C, PB and CB by determining $p_{ij}(a)$ and $c_i(a)$ for $a > 0$ for each case separately. As for the transition costs, they consist of production, holding and penalty costs, and it is convenient to define for $i \in \Omega$ and $a \in A(i)$:

$$\begin{aligned}
c_h^m(i) &:= \text{expected holding costs during a production run for model } m,\\
&\ \text{starting the run with an inventory of } i \text{ items} \quad (m = \mathrm{P, C, PB, CB});\\
c_p^m(i) &:= \text{expected penalty costs during a production run for model } m,\\
&\ \text{starting the run with an inventory of } i \text{ items} \quad (m = \mathrm{P, C});\\
c_p^m(i,a) &:= \text{expected penalty costs during a production run for } a \text{ items for}\\
&\ \text{model } m, \text{ starting the run with an inventory of } i \text{ items} \quad (m = \mathrm{PB, CB}).
\end{aligned}$$

Noting that only the penalty costs for models PB and CB (the case $D > 0$) depend on the batch size a, we have that

$$c_i(a) = \begin{cases} K + ca + c_h^m(i) + c_p^m(i) & \text{if } m = \mathrm{P, C} \\ K + ca + c_h^m(i) + c_p^m(i,a) & \text{if } m = \mathrm{PB, CB} \end{cases} \quad (i \in \Omega;\ a \in A(i)). \tag{5.12}$$

Once we have specified all ingredients for the SMDP, we can write down the optimality equations and use value iteration or policy iteration to find an optimal policy.

5.3 The case $D = 0$

The special case $D = 0$ corresponds to an inventory model in which demand that cannot be satisfied from stock on hand is lost (i.e., lost sales), constant lead time L and at most one outstanding order. The continuous-review version of this model has received some attention in the literature. In [Archibald 1981] an expression for the expected average costs per unit of time of an (s, S)-policy is derived for a compound-Poisson demand process, and the best (s, S)-policy for the lost-sales case is compared to the optimal (s, S)-policy for the backorder case. In [Johansen&Thorstenson 1993] a SMDP is constructed for a Poisson demand process and random lead times, and a special-purpose policy-iteration algorithm is developed with which optimal policies are computed for gamma-distributed lead times. In [Tijms 1986] (Example 3.5) a SMDP formulation is given for a similar production-inventory problem, assuming a Poisson demand process and a random production lead time that may depend on the lot size. For a survey of lost-sales inventory models see subsection 4.3.1.

5.3.1 Periodic review

Suppose that at the start of period 1 the process is in state i and action $a > 0$ is taken, i.e., on-hand inventory is i and a production run for a items is started. Then the on-hand inventory at the end of period L equals $(i - S_L)^+ + a$, and hence the transition probabilities for $a > 0$ are given by

$$p_{ij}(a) = \Pr\{(i - S_L)^+ + a = j\} = \begin{cases} \bar{Q}_i^{(L)} & \text{if } j = a \\ q_{i-j+a}^{(L)} & \text{if } a < j \leq i + a \\ 0 & \text{else} \end{cases} \quad (i \in \Omega). \tag{5.13}$$

Define

$l_n(i)$:= sum of expected holding and penalty costs incurred during n periods starting with an inventory of i items and barring any intermediate replenishments $(n = 1, 2, \ldots;\ i = 0, 1, \ldots)$,

then

$$c_h^{\mathrm{P}}(i) + c_p^{\mathrm{P}}(i) = l_L(i) \quad (i \in \Omega). \tag{5.14}$$

Obviously,

$$l_n(0) = pE\{S_n\} = pn\mu, \tag{5.15}$$

while for $i > 0$ we have that

$$\begin{aligned} l_n(i) &= E\Big\{\sum_{m=1}^{n} h(i - S_m)^+ + p(S_n - i)^+\Big\} \\ &= h \sum_{m=1}^{n} \sum_{k=0}^{i} (i - k) q_k^{(m)} + p \sum_{k=i+1}^{\infty} (k - i) q_k^{(n)} \\ &= h \sum_{m=1}^{n} \sum_{k=0}^{i-1} Q_k^{(m)} + p\Big(E\{S_n\} - \sum_{k=0}^{i-1} (1 - Q_k^{(n)})\Big) \\ &= h \sum_{m=1}^{n-1} \sum_{k=0}^{i-1} Q_k^{(m)} + (h + p) \sum_{k=0}^{i-1} Q_k^{(n)} + p(n\mu - i). \end{aligned} \tag{5.16}$$

For $n = 1$ (5.16) reduces to the one-period loss function $l(i)$ in (5.7). By conditioning on the demand in the next period, we obtain a recursive relation for $l_n(i)$:

$$l_n(i) = l(i) + \sum_{k=0}^{i-1} q_k l_{n-1}(i - k) + \bar{Q}_i l_{n-1}(0) \quad (n = 2, 3, \ldots, L;\ i = 0, 1, \ldots). \tag{5.17}$$

Using (5.4)–(5.6) and (5.11)–(5.14), the optimality equations can be written as

$$v(i) = \min_{a=0,1,2,\ldots} z(i, a) \quad (i \in \Omega), \tag{5.18}$$

with

$$z(i, a) := \begin{cases} l(i) - g + \bar{Q}_i v(0) + \sum_{j=1}^{i} q_{i-j} v(j) & \text{if } a = 0; \\ K + ca + l_L(i) - Lg + \bar{Q}_i^{(L)} v(a) + \sum_{j=a+1}^{i+a} q_{i-j+a}^{(L)} v(j) & \text{if } a > 0 \end{cases} \quad (i \in \Omega). \tag{5.19}$$

To formulate a value-iteration algorithm for a SMDP, a data transformation is needed to account for the non-identical transition times (see (1.15)). The idea is to add self-transitions in such a way that the "new" transition times becomes state- and action-independent, while the average sojourn time in any state i, given action a, is equal to the "old" transition time $\tau_i(a)$. For our SMDP this implies that in state i and for $a > 0$ the process remains in state i with probability $1-\frac{1}{L}$, while all other transition probabilities are reduced by a factor L. Consequently, the dynamic programming equations for the value-iteration algorithm can be written as

$$v_n(i) = \min_{a=0,1,2,\ldots} z_n(i,a) \quad (i \in \Omega), \tag{5.20}$$

with

$$z_n(i,a) := \begin{cases} l(i) + \bar{Q}_i v_{n-1}(0) + \sum_{j=1}^{i} q_{i-j} v_{n-1}(j) & \text{if } a = 0 \\ \frac{1}{L}\Big(K + ca + l_L(i) + \bar{Q}_i^{(L)} v_{n-1}(a) + \\ \quad \sum_{j=a+1}^{i+a} q_{i-j+a}^{(L)} v_{n-1}(j)\Big) + (1-\frac{1}{L}) v_{n-1}(i) & \text{if } a > 0 \end{cases} \quad (i \in \Omega). \tag{5.21}$$

5.3.2 Continuous review

Next we consider the case where demand follows a compound Poisson process. The (one-step) transition probabilities and costs for $a > 0$ are now more difficult to obtain, mainly because the number of demand arrival epochs during the production lead time is not constant but equal to the Poisson random variable $N(L)$. Obviously, the total demand during the production lead time equals $Y(L)$, and it follows that the transition probabilities for $a > 0$ are given by

$$p_{ij}(a) = \Pr\{(i - Y(L))^+ + a = j\} = \begin{cases} \bar{Q}_i(L) & \text{if } j = a \\ q_{i-j+a}(L) & \text{if } a < j \le i+a \\ 0 & \text{else} \end{cases} \quad (i \in \Omega). \tag{5.22}$$

The distribution of $Y(L)$ can only be computed explicitly for some special cases. In particular, if X_i $(i = 1, 2, \ldots)$ has a Poisson (geometric) distribution then $Y(L)$ has a Neyman type A (Pólya-Aeppli) distribution; see Appendix 5.A for details. However, for computational purposes the pmf of $Y(L)$ can in general be computed efficiently by using the well-known recursive scheme

$$\begin{aligned} q_0(t) &= e^{-\lambda t(1-q_0)}; \\ q_j(t) &= \frac{\lambda t}{j} \sum_{k=1}^{j} k q_k q_{j-k}(t) \quad (j = 1, 2, \ldots) \end{aligned} \tag{5.23}$$

(see e.g. [Adelson 1966] or [Tijms 1994], Theorem 1.2.6).

To compute $c_h^C(i)$, the expected holding costs during an interval of length L and given a starting inventory of i items, we condition on $N(L)$ and use the fact that

$$E\{A_i \mid N(L) = n\} = \frac{L}{n+1} \quad (i = 1, \ldots, n+1), \tag{5.24}$$

setting $A_{n+1} := L - B_n$ for convenience. Define $S_0 := 0$, $f_n := \Pr\{N(L) = n\}$ and $F_n := \sum_{i=0}^{n} f_i$, then we have that

$$\begin{aligned}
c_h^{\mathrm{C}}(i) &= E\Big\{h \sum_{k=1}^{N(L)+1} A_k(i - S_{k-1})^+\Big\} \\
&= h \sum_{n=0}^{\infty} f_n E\Big\{\sum_{k=1}^{n+1} A_k(i - S_{k-1})^+ \mid N(L) = n\Big\} \\
&= h \sum_{n=0}^{\infty} f_n \sum_{k=1}^{n+1} E\{A_k \mid N(L) = n\} E\{(i - S_{k-1})^+\} \\
&= h \sum_{n=0}^{\infty} e^{-\lambda L} \frac{(\lambda L)^n}{n!} \frac{L}{n+1} \sum_{k=1}^{n+1} \sum_{j=1}^{i-1} Q_j^{(k-1)} \\
&= \frac{h}{\lambda} \sum_{k=1}^{\infty} \sum_{n=k-1}^{\infty} e^{-\lambda L} \frac{(\lambda L)^{n+1}}{(n+1)!} \sum_{j=1}^{i-1} Q_j^{(k-1)} \\
&= \frac{h}{\lambda} \sum_{k=0}^{\infty} (1 - F_k) \sum_{j=1}^{i-1} Q_j^{(k)} \quad (i \in \Omega).
\end{aligned} \tag{5.25}$$

The expected penalty costs during the production run can be expressed in terms of the lead-time demand distribution, namely

$$c_p^{\mathrm{C}}(i) = E\{p(Y(L) - i)^+\} = p\Big(\lambda\mu - \sum_{k=1}^{i} \bar{Q}_k(L)\Big) \quad (i \in \Omega). \tag{5.26}$$

5.4 The case $0 < D \leq L$

5.4.1 Periodic review

Suppose that at the start of period 1 on-hand inventory is i and a production batch of a items is started. To compute on-hand inventory at the next decision epoch, being the end of period L, it is necessary to distinguish between periods $1, \ldots, L-D$ (the first $L-D$ periods) and periods $L-D+1, \ldots, L$ (the last D periods). Since demand in period n can be backordered until the delay-limit expires at the end of period $n+D-1$, demand in the first $L-D$ periods *cannot* be backordered from the incoming production batch at the end of period L, whereas demand in the last D periods *can*. Consequently, demand in the first $L-D$ periods exceeding the starting inventory i is lost, and the remaining inventory at the end of period $L-D$ is just $(i - S_{L-D})^+$. Demand in the last D periods is lost if it exceeds the intermediate inventory $(i - S_{L-D})^+$ plus the production batch a, and hence on-hand inventory at the end of period L (after backordering) is given by

$$\Big((i - S_{L-D})^+ + a - S_{L-D+1,L}\Big)^+ \tag{5.27}$$

(with $S_{m,n} := \sum_{i=m}^{n} X_i$). This leads to the following transition probabilities:

$$
\begin{aligned}
p_{ij}(a) &= \sum_{k=0}^{i-1} \Pr\{S_{L-D}=k, S_{L-D+1,L} \ge i-k+a\} + \Pr\{S_{L-D} \ge i, S_{L-D+1,L} \ge a\} \\
&= \sum_{k=0}^{i-1} q_k^{(L-D)} \bar{Q}_{i-k+a}^{(D)} + \bar{Q}_i^{(L-D)} \bar{Q}_a^{(D)} \quad (i \in \Omega;\ j=0); \\
p_{ij}(a) &= \sum_{k=0}^{i-1} \Pr\{S_{L-D}=k, S_{L-D+1,L} = i-k+a-j\} + \Pr\{S_{L-D} \ge i, S_{L-D+1,L} = a-j\} \\
&= \sum_{k=0}^{i-1} q_k^{(L-D)} q_{i-k+a-j}^{(D)} + \bar{Q}_i^{(L-D)} q_{a-j}^{(D)} \quad (i \in \Omega;\ 0 < j \le a); \qquad (5.28) \\
p_{ij}(a) &= \sum_{k=0}^{i+a-j} \Pr\{S_{L-D}=k, S_{L-D+1,L} = i-k+a-j\} \\
&= \sum_{k=0}^{i+a-j} q_k^{(L-D)} q_{i-k+a-j}^{(D)} \quad (i \in \Omega;\ a < j \le i+a).
\end{aligned}
$$

Next we turn to the one-step transition costs for $a > 0$,

$$
c_i(a) = K + ca + c_h^{\mathrm{PB}}(i) + c_p^{\mathrm{PB}}(i,a) \quad (i \in \Omega). \tag{5.29}
$$

Note that holding costs are only incurred for the starting inventory i, not for the production batch a. The inventory on hand at the end of period n is $(i - S_n)^+$ $(n = 1, \dots, L)$, and hence the expected holding costs during periods $1, \dots, L$ are given by

$$
c_h^{\mathrm{PB}}(i) = E\Big\{\sum_{n=1}^{L} h(i-S_n)^+\Big\} = h \sum_{n=1}^{L} E\{(i-S_n)^+\} = h \sum_{n=1}^{L} \sum_{k=0}^{i-1} Q_k^{(n)} \quad (i \in \Omega). \tag{5.30}
$$

As explained before, the penalty costs $c_p^{\mathrm{PB}}(i,a)$ consist of two parts corresponding to sales lost in the first $L-D$ periods and sales lost in the last D periods. Using similar reasoning it follows that the expected penalty costs in the first $L-D$ periods amount to

$$
E\{p(S_{L-D}-i)^+\} = p \sum_{k=i+1}^{\infty} (k-i) q_k^{(L-D)} = p\left((L-D)\mu - \sum_{k=1}^{i} \bar{Q}_k^{(L-D)}\right), \tag{5.31}
$$

while the expected penalty costs in the last D periods amount to

$$
\begin{aligned}
& E\Big\{p\big(S_{L-D+1,L} - (i - S_{L-D})^+ - a\big)^+\Big\} \\
&= \sum_{k=0}^{i-1} q_k^{(L-D)} E\{p(S_{L-D+1,L} - i + k - a)^+\} + \bar{Q}_i^{(L-D)} E\{p(S_{L-D+1,L} - a)^+\} \\
&= p\left(\sum_{k=0}^{i-1} q_k^{(L-D)}\Big(D\mu - \sum_{l=1}^{i-k+a} \bar{Q}_l^{(D)}\Big) + \bar{Q}_i^{(L-D)}\Big(D\mu - \sum_{l=1}^{a} \bar{Q}_l^{(D)}\Big)\right) \\
&= p\left(D\mu - \sum_{k=0}^{i-1} q_k^{(L-D)} \sum_{l=1}^{i-k+a} \bar{Q}_l^{(D)} - \bar{Q}_i^{(L-D)} \sum_{l=1}^{a} \bar{Q}_l^{(D)}\right). \qquad (5.32)
\end{aligned}
$$

Summing (5.31) and (5.32), we find that

$$c_p^{\mathrm{PB}}(i,a) = p\left(L\mu - \sum_{k=1}^{i} \bar{Q}_k^{(L-D)} - \sum_{k=0}^{i-1} q_k^{(L-D)} \sum_{l=1}^{i-k+a} \bar{Q}_l^{(D)} - \bar{Q}_i^{(L-D)} \sum_{l=1}^{a} \bar{Q}_l^{(D)}\right) \tag{5.33}$$

($i \in \Omega$; $a > 0$).

Remark. For $D = L$ the above formulas remain valid by setting $q_0^{(0)} := 1$. In this case (5.28) and (5.33) simplify to

$$p_{ij}(a) = \begin{cases} \bar{Q}_{i+a}^{(L)} & \text{if } j = 0 \\ q_{i+a-j}^{(L)} & \text{if } 0 < j \leq i+a \end{cases} \qquad (i \in \Omega;\ a > 0) \tag{5.34}$$

and

$$c_p^{\mathrm{PB}}(i,a) = p\left(L\mu - \sum_{k=1}^{i+a} \bar{Q}_k^{(L)}\right) \quad (i \in \Omega;\ a > 0), \tag{5.35}$$

respectively.

5.4.2 Continuous review

Analogously to model PB, on-hand inventory at the end of the production run is

$$\left((i - Y(L-D))^+ + a - Y(L-D, L)\right)^+ \tag{5.36}$$

(with $Y(t,u) := \sum_{i=N(t)+1}^{N(u)} X_i$), and the transition probabilities are given by

$$p_{ij}(a) = \begin{cases} \sum_{k=0}^{i-1} q_k(L-D)\bar{Q}_{i-k+a}(D) + \bar{Q}_i(L-D)\bar{Q}_a(D) & \text{if } j = 0; \\ \sum_{k=0}^{i-1} q_k(L-D)q_{i-k+a-j}(D) + \bar{Q}_i(L-D)q_{a-j}(D) & \text{if } 0 < j \leq a; \\ \sum_{k=0}^{i+a-j} q_k(L-D)q_{i-k+a-j}(D) & \text{if } a < j \leq i+a \end{cases} \tag{5.37}$$

($i \in \Omega$, cf. (5.28)). The expected holding costs are the same as for model C (see (5.25)):

$$c_h^{\mathrm{CB}}(i) = E\{h \sum_{k=1}^{N(L)+1} A_k(i - S_{k-1})^+\} = \frac{h}{\lambda} \sum_{k=0}^{\infty} (1 - F_k) \sum_{j=1}^{i-1} Q_j^{(k)} \quad (i \in \Omega). \tag{5.38}$$

Finally, the expected penalty costs follow directly from (5.33):

$$c_p^{\mathrm{CB}}(i,a) = pE\left\{\left(Y(L-D) - i\right)^+ + \left(Y(L-D,L) - (i - Y(L-D))^+ - a\right)^+\right\} =$$
$$p\left(\lambda L\mu - \sum_{k=1}^{i} \bar{Q}_k(L-D) - \sum_{k=0}^{i-1} q_k(L-D) \sum_{l=1}^{i-a+k} \bar{Q}_l(D) + \bar{Q}_i(L-D) \sum_{l=1}^{a} \bar{Q}_l(D)\right) \tag{5.39}$$

($i \in \Omega$; $a > 0$).

5.5 The complexity of the optimal policy

In this section we illustrate the complexity of the optimal policy by taking a closer look at model P with $L = 1$, the periodic-review lost-sales model with a lead time of one period. Note that the assumption of at most one production run at a time is automatically satisfied in this case. First we show how to compute the stationary distribution of on-hand inventory and the expected average costs for a general policy $\mathbf{R} := (R_1, \ldots, R_s)$, with R_i the batch size when on-hand inventory is i and $s := \max\{i : R_i > 0\}$. To this end, define

$$I_n(\mathbf{R}) \;:=\; \text{inventory on hand at the end of period } n \text{ under policy } \mathbf{R},$$

then clearly $\{I_n(\mathbf{R});\, n = 1, 2, \ldots\}$ is a Markov chain on the state space $\{0, \ldots, i_{\max}\}$ with $i_{\max} := \max\limits_{i=0,\ldots,s} \{i + R_i\}$. The transition probabilities are given by

$$p_{ij}(R_i) = \begin{cases} \bar{Q}_i & \text{if } j = R_i \\ q_{i+R_i-j} & \text{if } R_i < j \leq i + R_i \end{cases} \qquad (i = 0, \ldots, i_{\max}), \tag{5.40}$$

leading to the following balance equations for the stationary probabilities $\pi_i(\mathbf{R})$:

$$\pi_j(\mathbf{R}) = \sum_{i:R_i<j\leq i+R_i} \pi_i(\mathbf{R}) q_{i+R_i-j} + \sum_{i:R_i=j} \pi_i(\mathbf{R}) \bar{Q}_i \quad (j = 0, \ldots, i_{\max}). \tag{5.41}$$

Together with the normalization equation $\sum_i \pi_i(\mathbf{R}) = 1$ we can solve this system of equations numerically to obtain the stationary distribution of on-hand inventory at the start of a period. Unfortunately, due to the complex nature of the index sets, it is in general not possible to solve (5.41) in closed form.

Once we have the stationary distribution of on-hand inventory we can compute the expected average costs $g(\mathbf{R})$, by weighing the $\pi_i(\mathbf{R})$ $(i = 0, \ldots, i_{\max})$ with the one-period costs $c_i(R_i)$. Since

$$c_i(R_i) = K\delta(R_i) + cR_i + l(i), \tag{5.42}$$

(with $\delta(x) := 1$ if $x \leq 0$ and $\delta(x) := 0$ if $x > 0$) it follows that

$$g(\mathbf{R}) = K \sum_{i:R_i>0} \pi_i(\mathbf{R}) + c \sum_{i=0}^{i_{\max}} R_i \pi_i(\mathbf{R}) + \sum_{i=0}^{i_{\max}} l(i) \pi_i(\mathbf{R}). \tag{5.43}$$

Clearly $g(\mathbf{R})$ is a very complex function of $\mathbf{R}$, and the only convenient way of minimizing this function is to use an algorithm like policy iteration or value iteration (see section 1.5).

Regarding the optimal policy one would expect that R_i (batch size) is monotonically decreasing in i (on-hand inventory), but it turns out that this is not necessarily true; there are examples where the optimal policy is such that there exist i and j with $i > j$ and $R_i > R_j$. We illustrate this phenomenon for the following instance: X_n has a Poisson distribution with mean 10, and $K = 10$, $c = 0$ and $h = 1$. Table 5.1 gives the minimal expected average costs and the optimal policy for a range of values for p, as well as the computation time on a 486 PC of the value-iteration algorithm with an accuracy of 10^{-3}. Here n^m denotes a string of m n's, n- m denotes the string $n, n-1, \ldots, m$ $(m < n)$ and the remaining elements of $\mathbf{R}^*$ are zero; e.g., $(10^2, 9\text{-}7) = (10, 10, 9, 8, 7, 0, \ldots, 0)$.

p	g^*	$\mathbf{R}^*$	time
1	10	(0)	0:17
2	12.470	$(18^9, 17^2, 16^2, 15)$	3:46
3	13.756	$(20^7, 19^4, 18^2, 17, 16, 14)$	2:51
5	15.328	$(22^7, 21^4, 20^2, 19, 17\text{-}14)$	1:38
10	17.316	$(24^9, 23^3, 22, 12^2, 18\text{-}13)$	0:29
15	18.350	$(25^7, 15^2, 14^3, 13^2, 12^2, 18, 17^2, 16\text{-}14)$	0:20
20	19.040	$(15^{11}, 14^2, 13^2, 12^2, 18\text{-}13)$	0:19
25	19.575	$(16^7, 15^5, 14^2, 13^2, 12, 19\text{-}13)$	0:20
30	19.990	$(16^9, 15^4, 14^2, 13, 12^2, 18\text{-}13)$	0:19
35	20.356	$(16^{10}, 15^3, 14^2, 13^2, 12, 19\text{-}13)$	0:20
40	20.632	$(16^{11}, 15^3, 14^2, 13, 12, 19\text{-}13)$	0:20
45	20.901	$(17^6, 16^6, 15^2, 14^2, 13, 12^2, 18\text{-}13)$	0:19
50	21.152	$(17^8, 16^4, 15^2, 14^2, 13, 12^2, 19\text{-}14)$	0:19
60	21.509	$(17^{10}, 16^3, 15^2, 14^2, 13\text{-}11, 18\text{-}13)$	0:17
70	21.852	$(17^{11}, 16^3, 15, 14^2, 13\text{-}11, 18\text{-}13)$	0:17
80	22.134	$(17^{12}, 16^2, 15^2, 14, 13^2, 12, 11, 18\text{-}13)$	0:15
90	22.350	$(18^8, 17^4, 16^2, 15^2, 14^2, 13\text{-}10, 17\text{-}13)$	0:15
100	22.558	$(18^9, 17^4, 16^2, 15, 14^2, 13\text{-}8, 15\text{-}13)$	0:14
150	23.310	$(18^{12}, 17^2, 16^2, 15, 14^2, 13\text{-}5)$	0:14
200	23.869	$(19^9, 18^4, 17^2, 16^2, 15\text{-}4)$	0:14
250	24.223	$(19^{11}, 18^2, 17^2, 16^2, 15, 14^2, 13\text{-}5)$	0:14

Table 5.1: Non-monotonicity of the optimal policy ($D = 0$, $L = 1$)

It turns out that the optimal policy is non-monotone in a large region of p-values (indeed, if the expected average costs are computed for a monotone policy which is close to the non-monotone optimal policy, then these are found to be slightly higher). Moreover, it appears from all our numerical experiments that the optimal batch size R_i^* increases at most once as a function of i, in the form of a big "jump" from R_{i-1}^* to R_i^* (e.g., from 12 to 18 for $p \in \{10, 15, 20\}$ and from 12 to 19 for $p = 25$). Besides a big upward jump there can also be a big downward jump (e.g., from 22 to 12 for $p = 10$ and from 25 to 15 for $p = 15$).

A further investigation reveals that the jumps in R_i^* are caused by the presence of *two* local minima in the value function v_i (see also [de Rooij 1997], section 3.3.2). As an example we consider the case $p = 15$ (with $\lambda = 10$, $K = 10$, $c = 0$ and $h = 1$). Here the optimal value function v_i has two local minima in $i = 15$ and $i = 25$, respectively (see Figures 5.1 and 5.2). Since the optimality equations are given by

$$v(i) = \min_{a=0,1,\ldots} z(i, a) \quad (i = 0, 1, \ldots), \tag{5.44}$$

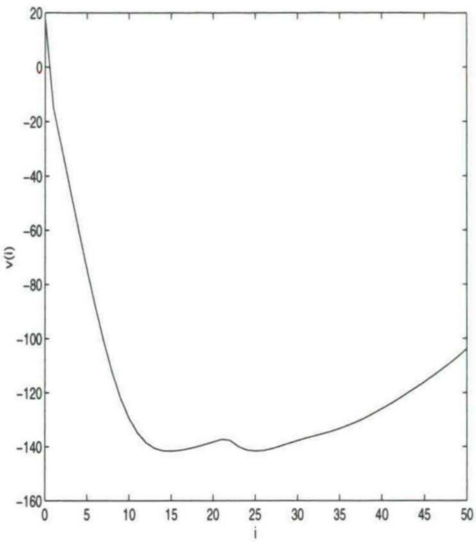

Figure 5.1: The value function $v(i)$ $(p = 15)$

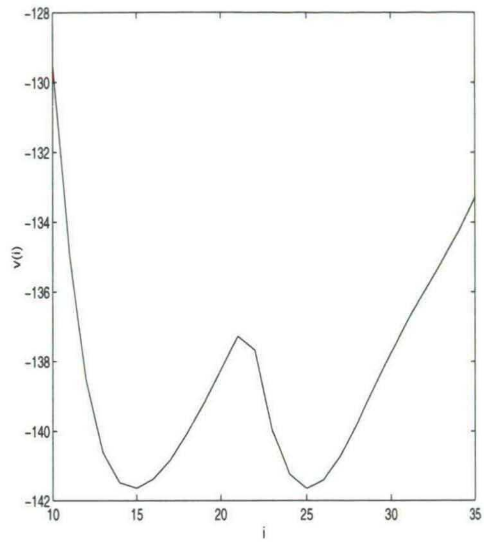

Figure 5.2: Magnification of $v(i)$ $(p = 15)$

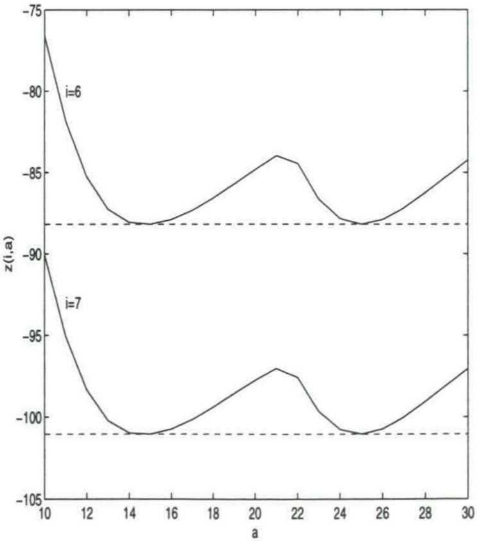

Figure 5.3: $R_6^* = 25$ while $R_7^* = 15$ $(p = 15)$

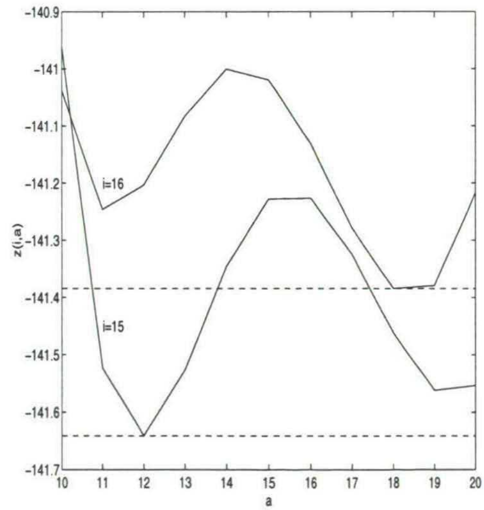

Figure 5.4: $R_{15}^* = 12$ while $R_{16}^* = 18$ $(p = 15)$

with

$$z(i,a) := \begin{cases} l(i) - g + \sum_{k=0}^{i-1} q_k v_{i-k} + \bar{Q}_i v(0) & \text{if } a = 0 \\ K + ca + l(i) - g + \sum_{k=0}^{i-1} q_k v_{i-k+a} + \bar{Q}_i v(a) & \text{if } a > 0 \end{cases} \qquad (i \geq 0) \tag{5.45}$$

(cf. also (5.18) and (5.19)), the minimizing action in state i is determined as $\operatorname{argmin}_a z(i,a)$. For fixed i the function $z(i,a)$ has the same shape as v_i, with two local minima, and the global minimum is attained in one of these local minima (see Figures 5.3 and 5.4). Now it is easily seen that a shift from the right to the left local minimum causes a downward jump, whereas a shift from the left to the right local minimum causes an upward jump. In our example, the minimum R_i^* shifts from 25 to 15 for $i = 7$ (see Figure 5.3), and from 12 to 18 for $i = 16$ (see Figure 5.4).

As a matter of fact, even if the optimal policy is nicely monotone and does not exhibit jumps, the value function is not necessarily unimodular. This makes it very difficult (if not impossible) to prove structural properties as we did for the discrete-time service model (see section 2.6). For example, it is intuitively obvious that $R_i = 0$ implies that $R_j = 0$ for $j > i$, but we were not able to prove this. Also, it remains to provide an economical explanation for the counterintuitive policies.

5.6 Analysis of a fixed policy

While a SMDP-formulation is primarily focused on finding optimal actions for all states $i \in \Omega$, it is often desirable to resort to restricted policies, i.e., policies that depend on a limited number of parameters. This is especially true if the optimal policy does not have a simple structure, and hence is not suited for practical purposes. As we have seen in the previous section, the optimal policy for the SMDP under consideration can be very complex and may even possess some counterintuitive properties, like non-monotonicity. Therefore we now consider some well-known simple policy subclasses, with the purpose of finding the best policy within these subclasses and comparing them with the global optimal policy. In the remainder of this section we present some general results that are useful when analysing a fixed policy $\mathbf{R}$. In section 5.7 we will extensively analyse the simple but effective (s,Q)-policy, while in section 5.8 we consider the class of (s,S,Q)-policies, a generalization of both the (s,Q)- and (s,S)-policy, that is often very close to optimality.

It is well-known that the expected average costs for any fixed policy $\mathbf{R}$ of a SMDP can be computed by solving the linear system

$$v_i(\mathbf{R}) = c_i(R_i) - g(\mathbf{R})\tau_i(R_i) + \sum_{j\in\Omega} p_{ij}(R_i)v_j(\mathbf{R}) \quad (i \in \Omega), \tag{5.46}$$

together with putting $v_k(\mathbf{R}) = 0$ for any $k \in \Omega$. This fact is also used in the value-determination step of the policy-iteration algorithm (see (1.13)).

An alternative approach uses the fact that any feasible policy for a SMDP induces an embedded Markov chain on decision epochs with transition matrix $\{p_{ij}(R_i);\ i,j \in \Omega\}$.

Consequently, we can compute the steady-state distribution at decision epochs $\{\pi_i(\mathbf{R}), i \in \Omega\}$ by solving the balance equations

$$\pi_j(\mathbf{R}) = \sum_{i \in \Omega} \pi_i(\mathbf{R}) p_{ij}(R_i) \quad (j \in \Omega), \tag{5.47}$$

and then compute the expected average costs $g(\mathbf{R})$ from

$$g(\mathbf{R}) = \frac{\sum_{i \in \Omega} \pi_i(\mathbf{R}) c_i(R_i)}{\sum_{i \in \Omega} \pi_i(\mathbf{R}) \tau_i(R_i)} \tag{5.48}$$

(see (1.9) and (1.10)).

Besides decision epochs it is also possible to use other embedding epochs, e.g., at the start or at the end of a production run, as long as the embedded process is a Markov chain. This is useful if one is interested in the stationary distribution of on-hand inventory at some special epochs other than decision epochs. An embedded Markov chain for policy $\mathbf{R}$ with embedded set E and state space Ω^E is characterized by transition probabilities $p_{ij}^E(R_i)$, expected transition times $\tau_i^E(R_i)$, and expected transition costs $c_i^E(R_i)$ $(i, j \in \Omega^E)$. Similar to (5.47) and (5.48), the steady-state distribution at embedding epochs $\{\pi_i^E(\mathbf{R}),\ i \in \Omega^E\}$ now follows from the balance equations

$$\pi_j^E(\mathbf{R}) = \sum_{i \in \Omega^E} \pi_i^E(\mathbf{R}) p_{ij}^E(R_i) \quad (j \in \Omega^E), \tag{5.49}$$

and the expected average costs from

$$g(\mathbf{R}) = \frac{\sum_{i \in \Omega^E} \pi_i^E(\mathbf{R}) c_i^E(R_i)}{\sum_{i \in \Omega^E} \pi_i^E(\mathbf{R}) \tau_i^E(R_i)}. \tag{5.50}$$

Clearly, $g(\mathbf{R})$ is independent of the choice of the embedding epochs. Choosing an embedded process is usually just a matter of balancing the complexity of the calculation of $\{\pi_i^E(\mathbf{R}),\ i \in \Omega^E\}$ against that of computing $\{c_i^E(R_i),\ i \in \Omega^E\}$ and $\{\tau_i^E(R_i),\ i \in \Omega^E\}$. Generally speaking, when Ω^E is chosen larger the calculation of the stationary distribution becomes more difficult and the calculation of the embedded sojourn- and cost functions becomes easier.

5.7 The (s, Q)-policy

In this section we focus on the (s, Q)-policy: start a production run for Q items at the first opportunity after on-hand inventory has dropped to or below s. Obviously,

$$R_i = \begin{cases} Q & \text{if } 0 \le i \le s; \\ 0 & \text{if } s < i \le s + Q. \end{cases} \tag{5.51}$$

We describe five different methods for the analysis of an (s, Q)-policy for model P, the periodic-review lost-sales model. Except for the last one, the same methods also apply to models C, PB and CB by using straightforward modifications.

(i) Value-determination

The most convenient way to compute the expected average costs for a given (s, Q)-policy is to solve system (5.46) for $\mathbf{R} = (Q, \ldots, Q, 0, \ldots, 0) =: (s, Q)$. Using (5.4)–(5.6) and (5.12)–(5.14), and setting $v_0(s, Q) = 0$, we obtain the following system of $s+Q+1$ equations:

$$\begin{aligned}
g &= p\mu + \frac{K + cQ + v_Q}{L}; \\
v_i &= K + cQ + l_L(i) - Lg + \bar{Q}_i^{(L)} v_Q + \sum_{j=Q+1}^{i+Q} q_{i+Q-j}^{(L)} v_j \quad (i = 1, \ldots, s); \\
v_i &= l(i) - g + \sum_{j=1}^{i} q_{i-j} v_j \quad (i = s+1, \ldots, s+Q)
\end{aligned} \tag{5.52}$$

(we write g and v_i instead of $g(s, Q)$ and $v_i(s, Q)$ for ease of notation).

(ii) Embedded Markov chain on decision epochs

Define

$$I_n^{E_1} := \text{on-hand inventory at the } n^{\text{th}} \text{ decision epoch} \quad (n = 1, 2, \ldots),$$

then $\{I_n^{E_1};\ n = 1, 2, \ldots\}$ is an embedded Markov chain with state space

$$\Omega^{E_1} := \{0, \ldots, s+Q\}. \tag{5.53}$$

The (one-step) transition probabilities, times and costs follow directly from the corresponding quantities of the SMDP by substituting $a = Q$ if $i \leq s$ and $a = 0$ if $i > s$:

$$p_{ij}^{E_1}(s, Q) = \begin{cases} \bar{Q}_i^{(L)} & \text{if } i \leq s,\ j = Q; \\ q_{i-j+Q}^{(L)} & \text{if } i \leq s,\ j > Q; \\ \bar{Q}_i & \text{if } i > s,\ j = 0; \\ q_{i-j} & \text{if } i > s,\ j \leq i. \end{cases} \tag{5.54}$$

$$\tau_i^{E_1}(s, Q) = \begin{cases} L & \text{if } i \leq s; \\ 1 & \text{if } i > s. \end{cases} \tag{5.55}$$

$$c_i^{E_1}(s, Q) = \begin{cases} K + cQ + l_L(i) & \text{if } i \leq s; \\ l(i) & \text{if } i > s. \end{cases} \tag{5.56}$$

Using (5.49) and (5.50), we see that the balance equations are given by

$$\begin{aligned}
\pi_0^{E_1} &= \sum_{i=s+1}^{s+Q} \pi_i^{E_2} \bar{Q}_i; \\
\pi_j^{E_1} &= \sum_{i=\max(s+1,j)}^{s+Q} \pi_i^{E_2} q_{i-j} \quad (j = 1, \ldots, Q-1); \\
\pi_Q^{E_1} &= \sum_{i=\max(s+1,Q)}^{s+Q} \pi_i^{E_2} q_{i-Q} + \sum_{i=0}^{s} \pi_i^{E_2} \bar{Q}_i^{(L)}; \\
\pi_j^{E_1} &= \sum_{i=\max(s+1,j)}^{s+Q} \pi_i^{E_2} q_{i-j} + \sum_{i=j-Q}^{s} \pi_i^{E_1} q_{i+Q-j}^{(L)} \quad (j = Q+1, \ldots, s+Q),
\end{aligned} \tag{5.57}$$

and the expected average costs by

$$g(s,Q)=\frac{\sum\limits_{i=0}^{s}\pi_i^{E_1}\left(K+cQ+l_L(i)\right)+\sum\limits_{i=s+1}^{s+Q}\pi_i^{E_1}l(i)}{L\sum\limits_{i=0}^{s}\pi_i^{E_1}+\sum\limits_{i=s+1}^{s+Q}\pi_i^{E_1}}. \tag{5.58}$$

(iii) Embedded Markov chain on epochs just before a production run is started

Define for $n=1,2,\ldots$

$$I_n^{E_2} \ := \ \text{on-hand inventory just before } n^{\text{th}} \text{ production run is started},$$

then $\{I_n^{E_2};\ n=1,2,\ldots\}$ is an embedded Markov chain with state space

$$\Omega^{E_2}:=\{0,\ldots,s\}. \tag{5.59}$$

Note that $s-I_n^{E_2}$ is the undershoot of inventory level s.

For this embedded Markov chain the expected transition times are no longer constant, but depend on the demand process; they require such quantities as the expected time that it takes to deplete a given inventory. This brings up the need for discrete renewal theory (or recurrent event theory), where the role of time is played by on-hand inventory. In section 4.6 we have discussed some important results from discrete renewal theory that we use in the sequel. In particular, the discrete renewal function and the forward recurrence time are defined by

$$N_i \ := \ \max\{n: S_n\le i\} \quad (i=0,1,\ldots); \tag{5.60}$$

$$\gamma_i \ := \ S_{N_i+1}-i \quad (i=0,1,\ldots), \tag{5.61}$$

respectively (see also (4.14) and (4.21)). The following two results are the starting point for our analysis.

Lemma 5.1 *Suppose that demand for an item arrives according to an i.i.d. process $\{X_n;\ n=1,2,\ldots\}$ with X_n the demand in period n, and that the inventory at the start of period 1 equals i (items). Then, in the absence of replenishments,*
(i) the number of periods until the starting inventory is depleted is equal to $N_{i-1}+1$, with N_i given by (5.60);
(ii) the undershoot of inventory level zero is equal to $\eta_i:=\gamma_{i-1}-1$, with γ_i given by (5.61).

Proof. (i) By (5.60), $N_{i-1}=n$ if and only if $S_n\le i-1$ and $S_{n+1}\ge i$, and hence inventory drops to zero in period $N_{i-1}+1$.
(ii) It follows from (i) that the undershoot of level zero (if any) occurs in period $N_{i-1}+1$, so that $\eta_i=S_{N_{i-1}+1}-i=\gamma_{i-1}-1$. □

Note that $\gamma_i>0$ by definition, while η_i is possibly zero (if $X_{N_{i-1}+1}=1$). Define

$$u_k^{(i)}:=\Pr\{\eta_i=k\},\quad U_k^{(i)}:=\sum_{j=0}^{k}u_j^{(i)},\quad \bar{U}_k^{(i)}:=\sum_{j=k}^{\infty}u_j^{(i)} \quad (i=0,1,\ldots;\ k=0,1,\ldots),$$

then it follows from Lemma 5.1(ii) and Theorem 4.5(i) that

$$u_k^{(i)} = \Pr\{\gamma_{i-1} = k+1\} = \frac{q_{i+k}}{1-q_0} + \sum_{j=1}^{i-1} q_{i+k-j} \sum_{n=1}^{\infty} q_j^{(n)} \quad (i \geq 0;\ k = 0, 1, \ldots). \tag{5.62}$$

Now, to determine the transition law of $\{I_n^{E_2}\}$, we need to distinguish between the cases $s < Q$ and $s \geq Q$. For $s < Q$ on-hand inventory upon completion of a run always exceeds s, so that there is at least one period between two consecutive runs. On the contrary, for $s \geq Q$ it is possible that on-hand inventory upon completion of a run does not exceed s, whence the next run is started immediately. Since the undershoot of level s starting with an inventory of i is equal to the undershoot of level 0 starting with an inventory of $s - i$, we have for $s < Q$ that

$$I_n^{E_2} = s - \eta_{(I_{n-1}^{E_2} - S_L)^+ + Q - s} \quad (n = 1, 2, \ldots), \tag{5.63}$$

and for $s \geq Q$ that

$$I_n^{E_2} = \begin{cases} s - \eta_{(I_{n-1} - S_L)^+ + Q - s} & \text{if } S_L < I_{n-1}^{E_2} - s + Q \\ (I_{n-1}^{E_2} - S_L)^+ + Q & \text{if } S_L \geq I_{n-1}^{E_2} - s + Q \end{cases} \quad (n = 1, 2, \ldots). \tag{5.64}$$

Starting with the case $s < Q$, the transition probabilities follow directly from (5.63):

$$p_{ij}^{E_2}(s, Q) = \begin{cases} \bar{U}_s^{(Q-s)} & \text{if } i = 0,\ j = 0; \\ u_{s-j}^{(Q-s)} & \text{if } i = 0,\ 0 < j \leq s; \\ \sum_{k=0}^{i-1} q_k^{(L)} \bar{U}_s^{(i-k+Q-s)} + \bar{Q}_i^{(L)} \bar{U}_s^{(Q-s)} & \text{if } 0 < i \leq s,\ j = 0; \\ \sum_{k=0}^{i-1} q_k^{(L)} u_{s-j}^{(i-k+Q-s)} + \bar{Q}_i^{(L)} u_{s-j}^{(Q-s)} & \text{if } 0 < i \leq s,\ 0 < j \leq s. \end{cases} \tag{5.65}$$

Conditioning on total demand during the production run we obtain

$$\tau_i^{E_2}(s, Q) = L + \sum_{k=0}^{i-1} q_k^{(L)} M_{i-k+Q-s-1} + \bar{Q}_i^{(L)} M_{Q-s-1} + 1 \quad (i \in \Omega^{E_2}). \tag{5.66}$$

To determine the expected transition costs we introduce the auxiliary function

H_i := sum of expected holding and penalty costs until the next production run is started, starting with an inventory of i items and barring any intermediate replenishments $(i = 0, 1, \ldots)$.

Obviously $H_i = 0$ for $0 \leq i \leq s$, while conditioning on demand in the next period yields the recursive relation

$$H_i = l(i) + \sum_{k=0}^{i-s-1} q_k H_{i-k} \quad (i = s+1, s+2, \ldots). \tag{5.67}$$

Using the transformation $\tilde{H}_i := H_{i+s+1}$ we can rewrite (5.67) as a standard discrete renewal equation

$$\tilde{H}_i = l(i+s+1) + \sum_{k=0}^{i} q_k \tilde{H}_{i-k} \quad (i = 0, 1, \ldots) \tag{5.68}$$

(see (4.19)). Applying Theorem 4.3 we find that

$$H_i = \frac{l(i)}{1-q_0} + \sum_{k=1}^{i-s-1} l(i-k) \sum_{n=1}^{\infty} q_k^{(n)} = \frac{l(i)}{1-q_0} + \sum_{k=1}^{i-s-1} l(i-k)\big(M_k - M_{k-1}\big) \quad (i > s). \tag{5.69}$$

Again we condition on total demand during the production run to obtain

$$c_i^{E_2}(s,Q) = K + cQ + l_L(i) + \sum_{k=0}^{i-1} q_k^{(L)} H_{i-k+Q} + \bar{Q}_i^{(L)} H_Q \quad (i \in \Omega^{E_2}). \tag{5.70}$$

Next we turn to the case $s \geq Q$, for which the transition probabilities follow from (5.64):

$$p_{ij}^{E_2}(s,Q) = \begin{cases} \bar{Q}_i^{(L)} & \text{if } 0 \leq i \leq s-Q,\ j = Q; \\ q_{i+Q-j}^{(L)} & \text{if } 0 \leq i \leq s-Q,\ Q < j \leq i+Q; \\ \displaystyle\sum_{k=0}^{i+Q-s-1} q_k^{(L)} \bar{U}_s^{(i-k+Q-s)} & \text{if } s-Q < i \leq s,\ j = 0; \\ \displaystyle\sum_{k=0}^{i+Q-s-1} q_k^{(L)} u_{s-j}^{(i-k+Q-s)} & \text{if } s-Q < i \leq s,\ 0 < j < Q; \\ \displaystyle\sum_{k=0}^{i+Q-s-1} q_k^{(L)} u_{s-j}^{(i-k+Q-s)} + \bar{Q}_i^{(L)} & \text{if } s-Q < i \leq s,\ j = Q; \\ \displaystyle\sum_{k=0}^{i+Q-s-1} q_k^{(L)} u_{s-j}^{(i-k+Q-s)} + q_{i+Q-j}^{(L)} & \text{if } s-Q < i \leq s,\ Q < j \leq s. \end{cases} \tag{5.71}$$

Similarly, it follows from (5.64) that

$$\tau_i^{E_2}(s,Q) = \begin{cases} L & \text{if } 0 \leq i \leq s-Q; \\ L + \displaystyle\sum_{k=0}^{i+Q-s-1} q_k^{(L)} (M_{i-k+Q-s-1} + 1) & \text{if } s-Q < i \leq s, \end{cases} \tag{5.72}$$

and

$$c_i^{E_2}(s,Q) = \begin{cases} K + cQ + l_L(i) & \text{if } 0 \leq i \leq s-Q; \\ K + cQ + l_L(i) + \displaystyle\sum_{k=0}^{i+Q-s-1} q_k^{(L)} H_{i-k+Q} & \text{if } s-Q < i \leq s. \end{cases} \tag{5.73}$$

(iv) Embedded Markov chain on epochs just after a production run is completed

Define for $n = 1, 2, \ldots$

$I_n^{E_3}$:= on-hand inventory just after the n^{th} production run is completed, then $\{I_n^{E_3};\ n=1,2,\ldots\}$ is an embedded Markov chain with state space

$$\Omega^{E_3} := \{Q,\ldots,s+Q\}. \tag{5.74}$$

Similar to method (iii) we distinguish between $s < Q$ and $s \geq Q$: for $s < Q$ we have that

$$I_n^{E_3} = (s - \eta_{I_{n-1}^{E_3}-s} - S_L)^+ + Q, \tag{5.75}$$

while for $s \geq Q$ we have that

$$I_n^{E_3} = \begin{cases} (I_{n-1}^{E_3} - S_L)^+ + Q & \text{if } I_{n-1}^{E_3} < s; \\ (s - \eta_{I_{n-1}^{E_3}-s} - S_L)^+ + Q & \text{if } I_{n-1}^{E_3} \geq s. \end{cases} \tag{5.76}$$

For $s < Q$ it follows from (5.75) that

$$p_{ij}^{E_3}(s,Q) = \begin{cases} \sum_{k=0}^{s-1} u_k^{(i-s)} \bar{Q}_{s-k}^{(L)} + \bar{U}_s^{(i-s)} & \text{if } Q \leq i \leq s+Q,\ j=Q; \\ \sum_{k=0}^{s-j+Q} u_k^{(i-s)} q_{s-k-j+Q}^{(L)} & \text{if } Q \leq i \leq s+Q,\ Q < j \leq s+Q, \end{cases} \tag{5.77}$$

and hence the balance equations are given by

$$\begin{aligned} \pi_Q^{E_3} &= \sum_{i=Q}^{s+Q} \pi_i^{E_3} \Big(\sum_{k=0}^{s-1} u_k^{(i-s)} \bar{Q}_{s-k}^{(L)} + \bar{U}_s^{(i-s)} \Big); \\ \pi_j^{E_3} &= \sum_{i=Q}^{s+Q} \pi_i^{E_3} \sum_{k=0}^{s-j+Q} u_k^{(i-s)} q_{s-k-j+Q}^{(L)} \quad (j = Q+1,\ldots,s+Q). \end{aligned} \tag{5.78}$$

Moreover,

$$\tau_i^{E_3}(s,Q) = M_{i-s-1} + L + 1, \tag{5.79}$$

and

$$c_i^{E_3}(s,Q) = K + cQ + H_i + \sum_{k=0}^{s-1} u_k^{(i-s)} l_L(s-k) + \bar{U}_s^{(i-s)} l_L(0). \tag{5.80}$$

For $s \geq Q$ it follows from (5.76) that

$$p_{ij}^{E_3}(s,Q) = \begin{cases} \bar{Q}_i^{(L)} & \text{if } Q \leq i \leq s,\ j=Q; \\ q_{i+Q-j}^{(L)} & \text{if } Q \leq i \leq s,\ Q < j \leq i+Q; \\ \sum_{k=0}^{s-1} u_k^{(i-s)} \bar{Q}_{s-k}^{(L)} + \bar{U}_s^{(i-s)} & \text{if } s < i \leq s+Q,\ j=Q; \\ \sum_{k=0}^{s-j+Q} u_k^{(i-s)} q_{s-k-j+Q}^{(L)} & \text{if } s < i \leq s+Q,\ Q < j \leq s+Q, \end{cases} \tag{5.81}$$

leading to the balance equations

$$\begin{aligned} \pi_Q^{E_3} &= \sum_{i=Q}^{s-1} \pi_i^{E_3} \bar{Q}_i^{(L)} + \sum_{i=s}^{s+Q} \pi_i^{E_3} \Big(\sum_{k=0}^{s-1} u_k^{(i-s)} \bar{Q}_{s-k}^{(L)}) + \bar{U}_s^{(i-s)} \Big); \\ \pi_i^{E_3} &= \sum_{i=\max\{j-Q,Q\}}^{s-1} \pi_i^{E_3} q_{i-j+Q}^{(L)} + \sum_{i=s}^{s+Q} \pi_j^{E_3} \sum_{k=0}^{s-j+Q} u_k^{(i-s)} q_{s-k-j+Q}^{(L)} \quad (j = Q+1,\ldots,s+Q). \end{aligned} \tag{5.82}$$

Also,

$$\tau_i^{E_3}(s,Q) = \begin{cases} L & \text{if } Q \le i \le s; \\ M_{i-s} + L + 1 & \text{if } s < i \le s + Q, \end{cases} \tag{5.83}$$

and

$$c_i^{E_3}(s,Q) = \begin{cases} K + cQ + l_L(i) & \text{if } Q \le i \le s; \\ K + cQ + H_i + \sum_{k=0}^{s-1} u_k^{(i-s)} l_L(s-k) + \bar{U}_s^{(i-s)} l_L(0) & \text{if } s < i \le s + Q. \end{cases} \tag{5.84}$$

(v) Discrete-time Markov chain on period endpoints

To find the stationary distribution of on-hand inventory at the end of an arbitrary period we can use a discrete-time Markov chain (see also the Remark in section 5.2). However, besides on-hand inventory, it is necessary to introduce an auxiliary state variable for the residual production time when a production run is underway (at least if $L > 1$). Therefore we define

I_n := inventory on hand at the end of period n;
T_n := number of periods until the ongoing production run (if any) is completed,

with the stipulation that $T_n := 0$ if no production run is underway. Then $\{(I_n, T_n);\ n = 1, 2, \ldots\}$ is a discrete-time Markov chain with state space

$$\Omega := \Big\{ (i,0) \mid i = 0, \ldots, s+Q \Big\} \bigcup \Big\{ (i,t) \mid i = 0, \ldots, s;\ t = 1, \ldots, L-1 \Big\}, \tag{5.85}$$

a total of $(L-1)(s+1) + Q$ states. It is important to note that it is not necessary to monitor the size of the production batch because every batch has the same size Q. If the batch sizes are not constant (e.g., for an (s, S)-policy) an additional state variable for the batch size is needed, which enlarges the state space dramatically.

Define the transition probabilities $p_{it,ju} := \Pr\{(I_n, T_n) = (j, u) \mid (I_{n-1}, T_{n-1}) = (i, t)\}$, then it is easily seen that

$$p_{it,ju}(s,Q) = \begin{cases} q_{i-j} & \text{if } i > s,\ 0 < j \le i,\ t = 0,\ u = 0; \\ \bar{Q}_i & \text{if } i > s,\ j = 0,\ t = 0,\ u = 0; \\ q_{i-j} & \text{if } i \le s,\ 0 < j \le i,\ t = 0,\ u = L-1; \\ \bar{Q}_i & \text{if } i \le s,\ j = 0,\ t = 0,\ u = L-1; \\ q_{i-j} & \text{if } 0 < j \le i,\ t > 0,\ u = t-1; \\ \bar{Q}_i & \text{if } j = 0,\ t > 0,\ u = t-1, \end{cases} \tag{5.86}$$

and hence the balance equations are given by

$$
\begin{aligned}
\pi_{00} &= \sum_{i=s+1}^{s+Q} \pi_{i0}\bar{Q}_i; \\
\pi_{j0} &= \sum_{i=\max(s+1,j)}^{s+Q} \pi_{i0}q_{i-j} \quad (j=1,\ldots,Q-1); \\
\pi_{Q0} &= \sum_{i=\max(s+1,Q)}^{s+Q} \pi_{i0}q_{i-j} + \sum_{i=0}^{s} \pi_{i,L-1}\bar{Q}_i; \\
\pi_{j0} &= \sum_{i=\max(s+1,j)}^{s+Q} \pi_{i0}q_{i-j} + \sum_{i=j-Q}^{s} \pi_{i,L-1}q_{i+Q-j} \quad (j=Q+1,\ldots,s+Q); \qquad (5.87) \\
\pi_{0u} &= \pi_{0,u-1} + \sum_{i=1}^{s} \pi_{i,u-1}\bar{Q}_i \quad (u=1,\ldots,L-1); \\
\pi_{ju} &= \sum_{i=j}^{s} \pi_{i,u-1}q_{i-j} \quad (j=1,\ldots,s;\ u=1,\ldots,L-1).
\end{aligned}
$$

Given the π_{it} we can compute the expected average costs from

$$g(s,Q) = (K+cQ)\sum_{i=0}^{s}\pi_{i0} + \sum_{i=0}^{s+Q}\pi_{i0}l(i) + \sum_{i=0}^{s}l(i)\sum_{t=1}^{L-1}\pi_{it}. \qquad (5.88)$$

Unfortunately, this last method does not work for any of the other models. For models C and CB it is not possible to formulate a continuous-time Markov chain on customer arrival epochs because of the deterministic production time (it would be possible for exponential production times, though).

5.8 The (s,S,Q)-policy

We now introduce the (s,S,Q)-policy, a policy that generalizes both the (s,Q)- and the (s,S)-policy. Under this policy the production quantity as a function of on-hand inventory is given by

$$R_i = \begin{cases} Q & \text{if } 0 \leq i \leq S-Q; \\ S-i & \text{if } S-Q < i \leq s; \\ 0 & \text{if } s < i \leq S. \end{cases} \qquad (5.89)$$

The (s,S,Q)-policy is only defined for $\max\{s,Q\} \leq S \leq s+Q$, while the maximum stock level is S. The two extreme cases $S=Q$ and $S=s+Q$ correspond to the (s,S)- and (s,Q)-policy, respectively. Numerical experience suggests that in some cases the global optimal policy is of this type, while in most other cases the optimal (s,S,Q)-policy is very close to the global optimal policy and clearly better than the (s,Q)-policy (see section 5.11). Surprisingly though, this policy has not yet been studied in the literature to the best of our knowledge.

Again, the expected average costs for this policy are easily computed by solving (5.46) for $\mathbf{R} = (Q,\ldots,Q,Q-1,\ldots,S-s,0,\ldots,0) =: (s,S,Q)$. For example, for model P this leads to the system

$$
\begin{aligned}
g &= p\mu + \frac{K + cQ + v_Q}{L}; \\
v_i &= K + cQ + l_L(i) - Lg + \bar{Q}_i^{(L)} v_Q + \sum_{j=Q+1}^{i+Q} q_{i+Q-j}^{(L)} v_j \quad (i = 1, \ldots, S-Q); \\
v_i &= K + c(S-i) + l_L(i) - Lg + \bar{Q}_i^{(L)} v_{S-i} + \sum_{j=S-i+1}^{S} q_{S-j}^{(L)} v_j \quad (i = S-Q+1, \ldots, s); \\
v_i &= l(i) - g + \sum_{j=1}^{i} q_{i-j} v_j \quad (i = s+1, \ldots, S)
\end{aligned}
\tag{5.90}
$$

(we write g and v_i instead of $g(s, S, Q)$ and $v_i(s, S, Q)$ for ease of notation).

The same applies to models C, PB and CB by substituting the appropriate quantities of the SMDP into (5.46).

5.9 The critical value for p

Clearly, if the penalty cost is sufficiently small then it will be optimal not to produce at all. As was noted earlier, the expected average cost for this policy is just

$$g(0, \ldots, 0) = p\mu. \tag{5.91}$$

In this section we show how the critical value $\bar{p}$, i.e., the value of p satisfying $\mathbf{R}^* = (0, \ldots, 0)$ if $p \leq \bar{p}$, can be computed numerically for the periodic-review lost-sales model (model P). For the case of geometric demand we derive a closed-form expression for $\bar{p}$.

To determine the critical value of p we compare the expected average costs of a no-production policy with the expected average costs of the best $(0, Q)$-policy. Numerical experience indicates that $\bar{p}$ is the value of p for which

$$\min_{Q=1,2,\ldots} g(0, Q) = p\mu. \tag{5.92}$$

Therefore, consider a fixed $(0, Q)$-policy: start a production run for Q items whenever on-hand inventory is depleted. Under this policy the on-hand inventory process is regenerative at epochs that a production run is started (or completed). A cycle consists of two phases: first the time until the starting inventory of Q items is depleted (equal to $N_{Q-1}+1$; see Lemma 5.1(i)), next the production lead time that brings on-hand inventory back to Q (equal to L). The holding costs during a cycle are given by $\sum_{n=1}^{\infty} h(Q - S_n)^+$, while penalty costs are incurred for the undershoot of the zero level (equal to $\eta_Q := \gamma_{Q-1} - 1$; see Lemma 5.1(ii)) and for all demand during the production lead time (equal to S_L). Consequently, we find upon application of the Renewal Reward Theorem that

$$g(0, Q) = \frac{K + cQ + h \sum_{n=1}^{\infty} E\{(Q - S_n)^+\} + p\big(E\{\gamma_{Q-1} - 1\} + L\mu\big)}{M_{Q-1} + L + 1}. \tag{5.93}$$

Suppose now that demand in a period is geometrically distributed with parameter r, i.e., $X_n \sim \mathrm{G}(r)$ and

$$q_k = \Pr\{X_n = k\} = (1-r)^k r \quad (k = 0, 1, \ldots); \quad \mu = \frac{1-r}{r}. \tag{5.94}$$

Then, using (4.25) and (4.26), it follows that

$$E\Big\{\sum_{n=1}^{\infty}(Q-S_n)^+\Big\} = \sum_{n=1}^{\infty}\sum_{k=0}^{Q}(Q-k)q_k^{(n)} = \sum_{k=0}^{Q}(Q-k)\frac{r}{1-r} = \frac{Q(Q+1)}{2\mu}. \tag{5.95}$$

This simple expression can be explained by viewing $\{X_n\}$ as a compound renewal process with unit interrenewal times and a $\mathrm{G}(r)$ compounding distribution. By Theorem 4.8 (section 4.6) this process is equivalent to a discrete renewal process $\{A_k\}$ with A_k ($k = 1, 2, \ldots$) having a $\mathrm{G}(1-r)$ distribution, and hence

$$E\Big\{\sum_{n=1}^{\infty}(Q-S_n)^+\Big\} = E\Big\{\sum_{k=1}^{Q}(Q-k+1)A_k\Big\} = E\{A_1\}\sum_{k=1}^{Q}k = \frac{Q(Q+1)}{2\mu}. \tag{5.96}$$

Using Theorem 4.7 and (5.95), (5.93) reduces to

$$g(0,Q) = \frac{K + cQ + \dfrac{hQ(Q+1)}{2\mu} + p(L+1)\mu}{\dfrac{Q}{\mu} + L + 1}. \tag{5.97}$$

We need the following lemma.

Lemma 5.2 *Let f, g and h be arbitrary functions defined on $V \subset I\!R$, and let a and b be constants. If $h(x) > 0$ and $ah(x) > g(x)$ for $x \in V$, then*

$$\min_{x\in V}\frac{f(x)+yg(x)}{h(x)} = ay + b \iff y = \min_{x\in V}\frac{f(x)-bh(x)}{ah(x)-g(x)}.$$

Proof. First suppose that $\min_{x\in V}\frac{f(x)+yg(x)}{h(x)} = ay+b$, then

$$\frac{f(x)+yg(x)}{h(x)} \geq ay+b, \forall_{x\in V} \iff y \leq \frac{f(x)-bh(x)}{ah(x)-g(x)}, \forall_{x\in V} \iff y \leq \min_{x\in V}\frac{f(x)-bh(x)}{ah(x)-g(x)}.$$

Now note that equality holds for $x^* = \mathrm{argmin}_{x\in V}\frac{f(x)+yg(x)}{h(x)}$. The opposite implication is proved analogously. $\square$

Using (5.92) and Lemma 5.2 we find that $\bar{p}$ satisfies

$$\bar{p} = c + \min_Q\Big\{\frac{K}{Q} + \frac{h(Q+1)}{2\mu}\Big\} = c + \min\Big\{\frac{K}{\lfloor Q^*\rfloor} + \frac{h(\lfloor Q^*\rfloor+1)}{2\mu}, \frac{K}{\lceil Q^*\rceil} + \frac{h(\lceil Q^*\rceil+1)}{2\mu}\Big\}, \tag{5.98}$$

where

$$Q^* = \sqrt{\frac{2\mu K}{h}} \tag{5.99}$$

is just the well-known EOQ-formula. It is noteworthy that the same condition holds for a continuous-review model where demand is generated by a Poisson process with parameter μ (see [Johansen&Thorstenson 1993], section 4).

5.10 A note on the lost-sales model with unit-sized renewal demand

In this section we make a side-step to a continuous-review lost-sales model with unit-sized renewal demand, i.e., we assume that demand is generated by a renewal process $\{N(t), t \geq 0\}$. As for the continuous-review models C and CB, we define A_i as the time between the $(i-1)^{\text{th}}$ and i^{th} demand arrival and $B_i := \sum_{j=1}^{i} A_j$, whence $N(t) := \max\{i : B_i \leq t\}$. Moreover, the forward recurrence time is now defined as $\gamma_t := S_{N(t)+1} - t$.

In [Hadley&Whitin 1963] an expression for the expected average costs per unit of time is derived for the case of a Poisson demand process and a constant lead time L. In [Johansen&Thorstenson 1993] (Appendix A) this analysis is extended to the case where the lead time L is a random variable. We now further generalize these results to the case where demand is generated by a renewal process, and as a byproduct we obtain the expressions in [Hadley&Whitin 1963] and [Johansen&Thorstenson 1993] in a more insightful way.

For a given (s, Q)-policy, define I_n as on-hand inventory just after the n^{th} replenishment order has arrived and let an order cycle be the time between two consecutive placements of a replenishment order. Note that the number of unit-sized demands in an order cycle that can be satisfied (necessarily from stock on hand) is exactly equal to Q. In analysing a (s, Q)-policy we have to distinguish between the cases $s < Q$, $s = Q$ and $s > Q$.

- *Case I: $s < Q$.*
  If $s < Q$ then $I_n \geq Q > s$ for all n, and hence an order will never be placed immediately after the previous order has arrived. Any order is placed on a demand arrival epoch that reduces on-hand inventory to s, and since demand is unit-sized no undershoot can occur. Consequently, the epochs at which an order is placed are regeneration epochs for the process $\{I_n\}$ and the expected average costs per unit of time can be found by a simple application of the Renewal Reward Theorem.

- *Case II: $s = Q$.*
  If $s = Q$ then possibly $I_n = Q = s$, in which case the $(n+1)^{\text{th}}$ order is placed immediately after the n^{th} order has arrived. Such an ordering epoch still causes no undershoot of the reorder level, but is not a demand arrival epoch. Therefore, these ordering epochs are only regeneration epochs if $\{N(t)\}$ is a Poisson process.

- *Case III: $s > Q$.*
  If $s > Q$ it may happen that $Q \leq I_n < s$, in which case the $(n+1)^{\text{th}}$ order is again placed immediately after arrival of the n^{th} order, while in addition there is an undershoot of the reorder level. So now, even in case of a Poisson demand process, these ordering epochs are no longer regeneration epochs. However, we can construct a regenerative process by defining a cycle as the time between two consecutive *demand arrival epochs* that reduce on-hand inventory to s. A cycle may now consist of a number of order cycles, where during the last order cycle in a regenerative cycle on-hand inventory is always positive.

We only consider case I, i.e., suppose that $s < Q$. Then the inventory on hand at the start of an order cycle (and at the end of an order cycle) is exactly equal to s, and the order cycle starts at a renewal epoch. For a given (s, Q)-policy, define

$$\begin{aligned} l(s,Q) &:= \text{expected length of one order cycle;} \\ c_h(s,Q) &:= \text{expected holding costs in one order cycle;} \\ c_p(s,Q) &:= \text{expected penalty costs in one order cycle.} \end{aligned}$$

As argued earlier, the ordering epochs are regeneration epochs, and hence it follows from the Renewal Reward Theorem that

$$g(s,Q) = \frac{K + cQ + c_h(s,Q) + c_p(s,Q)}{l(s,Q)}. \tag{5.100}$$

Hence we can restrict our attention to one order cycle and it is convenient to partition this cycle into three disjoint intervals: the first interval being the time until the starting inventory of s is depleted, the second interval the time (possibly zero) during which inventory on hand is zero and lost sales may occur, and the third interval the time (after arrival of the replenishment order) during which inventory on hand drops from Q to s. Now it is easily seen that the expected cycle length equals

$$l(s,Q) = E\Big\{B_s + (L - B_s)^+ + \sum_{i=1}^{Q-s} A_{s+(N(L)-s)^++i}\Big\} = E\{B_Q\} + E\{(L - B_s)^+\}, \tag{5.101}$$

since the A_i are i.i.d. random variables. Now define, as in [Johansen&Thorstenson 1993],

$$U(i) := E\{(N(L)–i)^+\} = \sum_{n=i}^{\infty} \Pr\{N(L) > n\} = E\{N(L)\}–i+\sum_{n=0}^{i-1} \Pr\{N(L) \leq n\}, \tag{5.102}$$

so that

$$c_p(s,Q) = pE\{(N(L) - s)^+\} = pU(s). \tag{5.103}$$

Since

$$\begin{aligned} E\{(B_s - L)^+\} &= \sum_{k=0}^{s-1} E\Big\{I_{\{N(L)=k\}}\Big(\gamma_L + \sum_{i=k+2}^{s} A_i\Big)\Big\} \\ &= E\{I_{\{N(L)<s\}}\gamma_L\} + E\{A_1\}\sum_{k=0}^{s-1} \Pr\{N(L) = k\}(s - k - 1) \\ &= E\{I_{\{N(L)<s\}}\gamma_L\} + E\{A_1\}\sum_{k=0}^{s-2} \Pr\{N(L) \leq k\} \\ &= E\{I_{\{N(L)<s\}}\gamma_L\} + E\{A_1\}\big(s - 1 - E\{N(L)\} + U(s-1)\big), \end{aligned} \tag{5.104}$$

we find for the holding costs incurred in a cycle that

$$\begin{aligned} c_h(s,Q) &= hE\Big\{\sum_{i=1}^{s}(s - i + 1)A_i + Q(B_s - L)^+ + \sum_{i=1}^{Q-s}(Q - i + 1)A_{i+(N(L)-s)^+}\Big\} \\ &= hE\Big\{\Big(\sum_{i=1}^{s} i + \sum_{i=s+1}^{Q} i\Big)A_1 + Q(B_s - L)^+\Big\} \\ &= hQE\{A_1\}\Big(s-1+\frac{Q+1}{2}-E\{N(L)\}+U(s-1)\Big) + hQ\, E\{I_{\{N(L)<s\}}\gamma_L\} \end{aligned} \tag{5.105}$$

Moreover, it follows from (5.101), using

$$E\{(L-B_s)^+\} = E\{L-B_s+(B_s-L)^+\} = E\{L\} - sE\{A_1\} + E\{(B_s-L)^+\}, \quad (5.106)$$

that

$$l(s,Q) = E\{L\} + E\{A_1\}\big(Q - E\{N(L)\} - 1 + U(s-1)\big) + E\{I_{\{N(L)<s\}}\gamma_L\}. \quad (5.107)$$

In case $\{N(t)\}$ is a Poisson process with parameter λ, we have that

$$E\{A_1\} = \frac{1}{\lambda}, \quad E\{I_{\{N(L)<s\}}\gamma_L\} = \frac{1}{\lambda}\Pr\{N(L)<s\}, \quad E\{N(L)\} = \lambda E\{L\}. \quad (5.108)$$

Consequently, (5.105) reduces to

$$c_h(s,Q) = \frac{hQ}{\lambda}\Big(s + \frac{Q+1}{2} - \lambda E\{L\} + U(s)\Big), \quad (5.109)$$

and (5.107) to

$$l(s,Q) = \frac{Q}{\lambda} + U(s), \quad (5.110)$$

in accordance with equation (A.9) in [Johansen&Thorstenson 1993].

5.11 Numerical comparisons

In this section we present numerical comparisons for model P and model PB. We consider various combinations of the delay-limit D and the lead time L with $D \le L$; the case $D=0$ corresponds to model P, while the case $0 < D \le L$ corresponds to model PB. For any (D,L) pair we vary $\mu \in \{5,10,15\}$, $K \in \{10,50\}$ and $p \in \{5,10\}$, setting $c=0$ and $h=1$. Moreover, we consider three different demand distributions (with mean μ and coefficient of variation c_X):

- $X_n \sim$ Poisson(γ) with $\mu = \gamma$ and $c_X = 1$;
- $X_n \sim$ Geometric(r) with $\mu = \frac{1-r}{r}$ and $c_X = \frac{1}{\sqrt{1-r}}$;
- "Two-point", i.e., $\Pr\{X_n = 1\} = \Pr\{X_n = k_{\max}\} = \frac{1}{2}$, $\mu = \frac{1+k_{\max}}{2}$ and $c_X = \frac{k_{\max}-1}{k_{\max}+1}$.

Tables 5.2, 5.3 and 5.4 compare the optimal policy with the best (s,Q)- and (s,S,Q)-policy for a Poisson, geometric and two-point demand distribution, respectively. The shorthand notation for the optimal policy $\mathbf{R}^* = (R_0^*, R_1^*, \ldots)$ is as follows: n^m denotes a string of m n's, n-m denotes the string $n, n-1, \ldots, m$ ($m < n$), $[n\text{-}m]^{i_n,\ldots,i_m}$ denotes $(n^{i_n}, \ldots, m^{i_m})$ ($m < n$), and the remaining elements of $\mathbf{R}^*$ are zero; e.g., $([10\text{-}8]^{2,2,1}, 7\text{-}5) = (10,10,9,9,8,7,6,5,0,\ldots,0)$.

We have already seen in section 5.5 that the optimal policy lacks a specific structure and may even possess counterintuitive properties, and this is confirmed by Tables 5.2-5.4. Some optimal policies have jumps ($R_i < R_{i-1} - 1$) or non-monotonicities ($R_i > R_{i-1}$), and these are indicated by boldface. However, the optimal policy for a two-point demand

D	L	μ	K	p	$g(\mathbf{R}^*)$	$\mathbf{R}^*$	$g(s^*, Q^*)$	$g(s^*, S^*, Q^*)$
0	1	5	10	5	10.8528	$(12^6, 11^2, 10)$	10.8898 (8,11)	10.8577 (8,18,12)
				10	12.2884	$(14^4, 13^3, 12^2, 11, \mathbf{9})$	12.3812 (10,12)	12.2911 (10,19,13)
			50	5	21.1844	(23^5)	21.1844 (4,23)	21.1844 (4,27,23)
				10	23.0695	$(25^5, 24^2, 23, 22)$	23.0954 (8,24)	23.0713 (8,30,25)
		10	10	5	15.3279	$(22^7, 21^4, 20^2, 19, \mathbf{17}\text{-}14)$	15.5469 (16,20)	15.3424 (17,32,21)
				10	17.3163	$(24^9, 23^2, 22, \mathbf{12^2}, \mathbf{18}\text{-}13)$	17.4588 (20,14)	17.3391 (20,34,23)
			50	5	30.1804	$(34^{10}, 33^2, 32, 31)$	30.1945 (13,33)	30.1814 (13,44,34)
				10	32.6007	$(36^9, 35^3, 34, 33^2, 32\text{-}30)$	32.6603 (17,34)	32.6040 (17,47,35)
		20	10	5	18.7496	$(23^{20}, 22^3, 21, 20^2, 19\text{-}8)$	19.8578 (35,20)	18.7708 (37,45,22)
				10	20.9803	$(25^{20}, 24^3, 23^2, 22, 21^2, 20\text{-}7)$	22.9793 (39,21)	21.0169 (41,48,23)
			50	5	42.4321	$(42^{20}, 41^3, 40^2, 39, 38^2, 37\text{-}34)$	42.5276 (31,41)	42.4640 (31,64,41)
				10	45.4999	$([45\text{-}40]^{20,3,1,2,1,2}, 39\text{-}37, \mathbf{35}\text{-}31)$	45.8508 (36,43)	45.5198 (36,67,44)
0	3	5	10	5	11.7816	$(14^{15}, 13^3)$	11.8006 (17,14)	11.7833 (17,29,14)
				10	13.9518	$(17^{13}, 16^4, 15^2, 14, 13)$	14.0206 (20,15)	13.9616 (20,33,16)
			50	5	21.4832	(23^{14})	21.4832 (13,23)	21.4832 (13,36,23)
				10	24.1170	$(26^{14}, 25^2, 24\text{-}22)$	24.1395 (18,24)	24.1186 (18,40,26)
		10	10	5	18.2666	$(29^{26}, 28^4, 27^2, 26^2, 25\text{-}23)$	18.3440 (35,28)	18.2746 (36,59,28)
				10	21.4713	$(32^{29}, 31^2, 30^3, 29, 28^2, 27\text{-}24)$	21.6617 (39,30)	21.4972 (40,64,31)
			50	5	31.0991	$(37^{28}, 36^3, \mathbf{29^3})$	31.1578 (33,30)	31.1572 (33,62,30)
				10	34.1093	$(33^{27}, 32^8, 31^2, 30)$	34.1179 (37,32)	34.1118 (37,67,32)
0	5	5	10	5	13.8799	$(21^{21}, 20^3, 19^2, 18)$	13.8873 (26,20)	13.8859 (26,44,20)
				10	17.0177	$(25^{21}, 24^4, 23^3, 22, 21)$	17.0846 (28,24)	17.0260 (29,50,24)
			50	5	21.6842	(24^{22})	21.6842 (21,24)	21.6842 (21,45,24)
				10	24.7035	$(25^{26}, 24^2)$	24.7120 (27,25)	24.7064 (27,51,25)
1	3	5	10	5	8.2872	$(16^{10}, 15^2, 14^2, 13)$	8.3245 (13,15)	8.3007 (13,26,16)
				10	9.7085	$(18^9, 17^3, 16^2, 15, 14)$	9.8081 (15,16)	9.7155 (15,29,17)
			50	5	17.8569	$(25^8, 24^3)$	17.8607 (10,24)	17.8583 (10,33,25)
				10	19.5789	$(27^6, 26^4, 25^2, 24, 23)$	19.5999 (13,25)	19.5800 (13,36,26)
		10	10	5	11.3398	$(31^{19}, 30^3, 29^2, 28\text{-}24)$	11.5168 (27,29)	11.3598 (28,52,31)
				10	13.3183	$(34^{18}, 33^3, 32^3, 31, 30, 29^2, 28\text{-}25)$	13.6467 (30,31)	13.3355 (31,55,33)
			50	5	23.6649	$(39^{18}, 38^2, 37\text{-}32)$	23.7211 (25,36)	23.6656 (25,57,39)
				10	25.6345	$(35^{24}, 34^2, 33^2, 32, 31)$	25.6664 (29,34)	25.6378 (29,60,35)
2	3	5	10	5	5.6545	$(19^3, 18^3, 17^2, 16, 15)$	5.7197 (9,17)	5.6578 (9,24,18)
				10	6.3924	$(20^5, 19^2, 18, 17, 16^2, 15)$	6.5536 (10,18)	6.3974 (11,25,20)
			50	5	14.7318	$(27^4, 26^2, 25)$	14.7361 (6,26)	14.7330 (6,31,27)
				10	15.7549	$(28^4, 27^2, 26, 25^2, 24)$	15.7893 (9,26)	15.7568 (9,33,27)
		10	10	5	6.6324	$(35^7, 34^4, 33^2, 32\text{-}25)$	6.8920 (19,32)	6.6348 (20,45,34)
				10	7.5494	$(37^7, 36^4, 35^2, 34, 33, 32^2, 31\text{-}26)$	7.9754 (21,33)	7.5590 (22,48,36)
			50	5	18.0146	$(43^7, 42^3, 41^2, 40\text{-}38, 37^2)$	18.0647 (16,40)	18.0152 (16,52,42)
				10	19.0965	$(44^8, 43^3, 42\text{-}40, 39^2, 38\text{-}35)$	19.1931 (19,39)	19.0986 (19,54,43)
2	5	5	10	5	7.7750	$(24^{15}, 23^2, 22, 21^2)$	7.8236 (18,23)	7.7796 (19,39,24)
				10	9.4695	$(27^{14}, 26^3, 25^2, 24\text{-}22)$	9.5834 (20,25)	9.4843 (21,43,26)
			50	5	15.2260	$(27^{14}, 26^2, 25)$	15.2347 (16,26)	15.2272 (16,41,27)
				10	16.7994	$(29^{14}, 28^3, 27^2, 26)$	16.8342 (19,27)	16.8058 (19,45,28)
4	5	5	10	5	3.9321	$(29^4, 28^2, 27^2, 26\text{-}24)$	4.0213 (10,27)	3.9351 (10,34,28)
				10	4.5349	$(31^2, 30^4, 29^2, 28\text{-}24)$	4.6800 (11,28)	4.5367 (12,36,30)
			50	5	10.6819	$(33^4, 32^2, 31^2, 30)$	10.7021 (8,31)	10.6841 (8,38,32)
				10	11.2917	$(34^4, 33^2, 32^2, 31\text{-}29)$	11.3402 (10,31)	11.2927 (10,39,33)

Table 5.2: Numerical comparison of optimal, (s, Q)- and (s, S, Q)-policy (Poisson)

D	L	μ	K	p	$g(\mathbf{R}^*)$	$\mathbf{R}^*$	$g(s^*,Q^*)$	$g(s^*,S^*,Q^*)$
0	1	5	10	5	15.4430	$(15^2, 14^3, 13, 12^2)$	15.4914 (7,14)	15.4455 (7,18,15)
				10	19.8468	$(19^3, 18^2, 17^2, 16, 15, 14^2, \mathbf{12}, 11)$	20.1084 (10,17)	19.8488 (12,23,19)
			50	5	23.4900	(23^2)	23.4900 (1,23)	23.4900 (1,24,23)
				10	28.1969	$(28^2, 27^2, 26^2, 25, 24^2)$	28.2335 (7,26)	28.1993 (8,31,27)
		10	10	5	27.0795	$([27-20]^{1,4,2,2,2,1,2,1}, \mathbf{17}, 16, 15^2)$	27.4710 (19,17)	27.1555 (16,34,26)
				10	35.1227	$([35-30]^{2,3,2,2,2,1}, [\mathbf{25}-15]^{2,1,2,1,2,1,2,1,1,2,1})$	35.3651 (27,20)	35.2011 (27,41,24)
			50	5	37.4755	$(37^3, 36^3, 35^2, 34^2, 33)$	37.4894 (9,36)	37.4782 (9,42,37)
				10	46.1119	$([46-37]^{2,3,2,2,2,1,2,1,1,2}, 36-34)$	46.2681 (19,41)	46.1193 (20,53,45)
0	3	5	10	5	16.6763	$(14^3, 13^{10}, 12^3)$	16.6874 (15,12)	16.6858 (15,26,13)
				10	22.2460	$(18^{12}, 17^3, 16^3, 15^2, 14^2, 13)$	22.3077 (21,16)	22.2652 (22,35,16)
			50	5	23.7472	(23^7)	23.7472 (6,23)	23.7472 (6,29,23)
				10	30.1654	$(30^5, 29^5, 28^3, 27^2, 26, 25)$	30.1976 (16,27)	30.1708 (16,41,29)
		10	10	5	29.6594	$([27-20]^{7,10,5,4,3,2,2,2})$	29.7101 (34,23)	29.6730 (34,53,24)
				10	40.5171	$([36-24]^{14,6,5,3,3,2,3,2,2,2,1,2,2})$	40.7070 (44,30)	40.5633 (46,68,31)
			50	5	39.3220	$(39^{10}, 38^7, 37^3, 36^3, 35)$	39.3361 (23,37)	39.3266 (23,57,38)
				10	50.8065	$([50-43]^{14,4,4,2,2,2,2,1}, [\mathbf{37}-34]^{3,2,3,1})$	50.9525 (40,36)	50.9335 (40,72,37)
0	5	5	10	5	17.5495	$(17^{16}, 16^4, 15^3)$	17.5531 (22,16)	17.5517 (22,37,16)
				10	24.4743	$(24^{15}, 23^5, 22^4, 21^3, 20^2, 19^2)$	24.5305 (29,22)	24.4854 (30,48,22)
			50	5	23.8985	(23^{12})	23.8985 (11,23)	23.8985 (11,34,23)
				10	31.4634	$(31^{13}, 30^5, 29^3, 28^2, \mathbf{24}^3)$	31.5015 (26,24)	31.5008 (26,49,25)
1	3	5	10	5	13.5109	$(18^4, 17^3, 16^2, 15^2, 14^2, 13)$	13.5727 (12,16)	13.5188 (13,25,17)
				10	17.9407	$([22-14]^{6,2,2,2,1,2,2,1,1})$	18.1043 (17,18)	17.9654 (18,31,21)
			50	5	20.8440	$(25^6, 24)$	20.8451 (6,25)	20.8440 (6,30,25)
				10	25.6371	$(30^5, 29^3, 28^2, 27^2, 26, 25)$	25.6856 (13,28)	25.6436 (13,38,29)
		10	10	5	23.5758	$([33-22]^{5,5,3,3,2,2,2,2,2,2,1,1})$	23.7313 (28,27)	23.6106 (28,49,31)
				10	32.1143	$([41-26]^{9,4,2,2,3,2,2,2,2,2,1,2,1,2,1,1})$	32.4551 (36,33)	32.1680 (37,62,37)
			50	5	32.8146	$(42^8, 41^4, 40^3, 39^2, 38^2, 37, 36)$	32.8523 (20,40)	32.8222 (20,55,41)
				10	41.4960	$([51-41]^{7,4,3,2,2,2,2,1,2,1,2}, 40-37)$	41.6890 (31,43)	41.5105 (31,67,49)
2	3	5	10	5	10.5790	$(20^3, 19^2, 18^2, 17, 16, 15^2)$	10.6601 (9,18)	10.5823 (10,24,20)
				10	13.8319	$(24^4, 23, 22^2, 21, 20^2, 19-16)$	14.0537 (13,20)	13.8378 (13,29,24)
			50	5	17.9643	$(27^3, 26^2, 25)$	17.9737 (5,27)	17.9645 (5,30,27)
				10	21.3309	$(31^2, 30^3, 29, 28^2, 27-25)$	21.3885 (9,29)	21.3350 (10,35,31)
		10	10	5	17.9533	$([39-26]^{2,3,3,1,2,2,1,1,2,1,1,2,1,1})$	18.2032 (20,34)	17.9655 (22,47,38)
				10	24.1908	$([47-33]^{2,4,2,2,1,2,1,2,1,1,2,1,1,1,2}, 32-29)$	24.7251 (26,39)	24.2087 (28,56,46)
			50	5	26.7833	$(46^4, 45^3, 44^2, 43, 42^2, 41, 40^2, 39)$	26.8519 (15,43)	26.7881 (15,53,46)
				10	32.9282	$([53-40]^{5,2,2,2,1,2,1,1,2,1,1,1,2,1})$	33.1712 (22,47)	32.9377 (23,62,53)
2	5	5	10	5	12.5489	$(23^9, 22^3, 21^2, 20^2, 19^2)$	12.6027 (17,21)	12.5605 (17,35,22)
				10	17.2965	$(29^8, 28^4, 27^2, 26^2, 25^2, 24^2, 23, 22^2)$	17.4480 (22,25)	17.3184 (22,43,28)
			50	5	18.8026	$(28^9, 27^3, 26)$	18.8098 (11,28)	18.8040 (11,37,28)
				10	23.5556	$(34^4, 33^6, 32^2, 31^3, 30, 29^2, 28^2)$	23.6174 (19,30)	23.5642 (19,46,32)
4	5	5	10	5	8.3635	$(29^3, 28^2, 27^2, 26-24, 23^2)$	8.4556 (10,27)	8.3661 (11,33,29)
				10	11.1349	$(34^3, 33^2, 32^2, 31, 30, 29^2, 28-25)$	11.3489 (13,30)	11.1431 (14,39,34)
			50	5	14.2198	$(34^2, 33^2, 32^2, 31, 30)$	14.2425 (7,32)	14.2219 (7,37,33)
				10	16.8728	$(38^2, 37^2, 36^2, 35, 34^2, 33-31)$	16.9621 (11,35)	16.8767 (11,42,37)

Table 5.3: Numerical comparison of optimal, (s,Q)- and (s,S,Q)-policy (geometric)

D	L	μ	K	p	$g(\mathbf{R}^*)$	$\mathbf{R}^*$	$g(s^*, Q^*)$	$g(s^*, S^*, Q^*)$
0	1	5	10	5	13.3535	$(14^3, 13^3, 12^2, 11, 10)$	13.3978 (9,12)	13.3546 (9,19,13)
				10	14.3871	$(18^4, \mathbf{9}^2, \mathbf{13}\text{-}5)$	15.0508 (12,9)	14.3871 (14,19,13)
			50	5	22.4867	(21^2)	22.4867 (1,21)	22.4867 (1,22,21)
				10	25.5953	$(28^2, 27, 26^2, 25\text{-}23, 22)$	25.6589 (9,24)	25.5986 (9,30,27)
		10	10	5	23.4878	$(19^{20}, 18\text{-}13)$	23.5050 (22,19)	23.4878 (25,38,19)
				10	24.3871	$(19^{16}, \mathbf{23}\text{-}5)$	26.8261 (31,19)	24.3871 (34,39,23)
			50	5	35.5056	$(40^3, 39^2, 38^2, 37\text{-}35, \mathbf{33})$	35.5090 (8,40)	35.5057 (10,42,40)
				10	41.3276	$(38^{13}, \mathbf{39}^7, 38\text{-}35)$	41.4008 (21,38)	41.3371 (24,58,39)
0	3	5	10	5	14.8640	$(18^4, 17\text{-}15, \mathbf{16}\text{-}14, 13^2, 12, 11^7)$	14.8765 (19,12)	14.8738 (19,30,12)
				10	18.4196	$(19^{15}, 18, \mathbf{19}^4, 18\text{-}16)$	18.4661 (20,19)	18.4197 (22,38,19)
			50	5	22.8737	$(23^7, 22^3)$	22.8742 (9,23)	22.8737 (9,31,23)
				10	27.3889	$(29^{13}, \mathbf{27}^2, 26\text{-}24, 23^2)$	27.4373 (19,25)	27.3902 (19,41,29)
		10	10	5	28.2590	$(38^4, 37\text{-}23, 22^2, 21^{19})$	28.2590 (39,21)	28.2590 (39,60,21)
				10	36.6898	$(39^{26}, \mathbf{36}, \mathbf{34}\text{-}24, \mathbf{39}^2, 38\text{-}28)$	37.0198 (48,30)	37.0188 (47,78,32)
			50	5	37.6293	$(40^{23}, 39\text{-}36)$	37.6295 (24,40)	37.6293 (26,62,40)
				10	46.2933	$(39^{40}, 38\text{-}35)$	46.2988 (41,39)	46.2933 (43,78,39)
0	5	5	10	5	16.1778	$(20^{15}, 19^8, 18^2, 17)$	16.1956 (24,19)	16.1780 (25,42,19)
				10	21.3099	$([29\text{-}21]^{14,1,2,0,1,1,1,1,3}, [\mathbf{26}\text{-}19]^{1,1,0,1,1,2,1,1})$	21.3120 (30,21)	21.3107 (31,50,21)
			50	5	23.0301	(22^{16})	23.0301 (15,22)	23.0301 (15,37,22)
				10	28.6068	$(30^{24}, \mathbf{27}^2, 26\text{-}24)$	28.6692 (28,26)	28.6458 (28,51,28)
1	3	5	10	5	11.2472	$(19^{11}, 18\text{-}14)$	11.2927 (12,19)	11.2472 (15,29,19)
				10	13.5551	$(20^{11}, 19^2, 18\text{-}13)$	13.8732 (17,19)	13.5718 (18,31,20)
			50	5	19.6847	$(25^3, 24^5, 23^2)$	19.6868 (9,24)	19.6853 (9,32,24)
				10	22.5805	$(29^{12}, 28\text{-}26)$	22.6314 (13,29)	22.5805 (14,40,29)
		10	10	5	21.5804	$(39^{22}, 38, 37, \mathbf{35}\text{-}26)$	21.9889 (31,30)	21.9262 (33,59,33)
				10	26.6912	$(40^{21}, 39^2, 38\text{-}23)$	27.7706 (37,36)	26.6949 (38,61,40)
			50	5	30.5652	$(39^{22}, 38\text{-}36)$	30.5659 (22,39)	30.5652 (24,60,39)
				10	35.9888	$(41^{16}, [\mathbf{43}\text{-}37]^{2,2,2,1,3,1,1}, \mathbf{35}\text{-}30, \mathbf{28})$	36.3018 (34,40)	35.9948 (34,62,42)
2	3	5	10	5	8.1238	$(19^{10}, 18)$	8.1292 (9,19)	8.1238 (10,28,19)
				10	9.2981	$(20^4, 19^3, \mathbf{21}, 20^2, 19\text{-}15)$	9.5679 (12,19)	9.3063 (14,29,20)
			50	5	16.5220	$(30^2, 29^2, 28, 27, \mathbf{24})$	16.5523 (5,29)	16.5234 (6,31,30)
				10	18.2722	$(28^2, 27^7, \mathbf{28}, 27)$	18.2736 (10,27)	18.2736 (10,37,27)
		10	10	5	15.4548	$(39^4, 38\text{-}34, \mathbf{39}^{11}, 38\text{-}35)$	15.4593 (20,39)	15.4555 (23,58,39)
				10	17.5928	$(40^4, 39^{10}, \mathbf{44}\text{-}42, 41^2, 40\text{-}26)$	18.4654 (30,40)	17.6232 (33,59,41)
			50	5	24.8514	$(43^2, 42^3, 41^{12}, \mathbf{42}^2)$	24.8575 (18,42)	24.8575 (18,60,42)
				10	27.1362	$(58^3, 57^2, 56\text{-}53, \mathbf{51}, \mathbf{48}\text{-}43, 42^2, 41\text{-}30)$	27.8800 (26,40)	27.3691 (30,59,41)
2	5	5	10	5	10.5627	$(24^7, 23^3, 22^2, 21^8)$	10.5848 (19,21)	10.5777 (19,40,22)
				10	13.6097	$(29^{14}, 28, 27, \mathbf{25}\text{-}23, 22^2, 21, 20)$	13.7797 (20,29)	13.6136 (22,42,29)
			50	5	17.2678	$(30^4, 29^9, 28)$	17.2687 (13,29)	17.2678 (13,41,29)
				10	20.1645	(29^{20})	20.1645 (19,29)	20.1645 (19,48,29)
4	5	5	10	5	6.2370	$(29^{10}, 28, 27)$	6.2521 (10,29)	6.2370 (11,38,29)
				10	7.4005	$(31^4, 30^2, \mathbf{32}, 31, 30^2, 29\text{-}24)$	7.6132 (12,30)	7.4017 (15,39,31)
			50	5	12.5427	$(33^3, 32^3, 31^3, 30)$	12.5553 (9,32)	12.5472 (9,39,32)
				10	13.7013	$(38^2, 37^2, 36, 35, \mathbf{33}\text{-}27)$	14.0072 (12,30)	13.7052 (12,39,38)

Table 5.4: Numerical comparison of optimal, (s, Q)- and (s, S, Q)-policy (two-point)

distribution (Table 5.4) must be interpreted with care due to the presence of transient states (e.g., all states $i \leq s - k_{\max}$ with $s := \max\{i : R_i > 0\}$ are transient). Clearly, actions in transient states do not influence the expected average costs and hence are of limited significance. Notice that the so-called "unichain condition" (see e.g. [Tijms 1994], Assumption 3.2.1) is also satisfied for two-point demand, since state s can be reached from any other recurrent state.

Firstly, the difference in costs between the global optimal policy and the optimal (s, Q)- and (s, S, Q)-policy is less than 1% in most cases. The (s, S, Q)-policy has an excellent performance and is always very close to the optimal policy, while the (s, Q)-policy performs poorly in some cases, e.g., for $D = 0$, $L = 1$, $\mu = 20$, $K = 10$ and Poisson demand it loses 5.9% and 9.5% for $p = 5$ and $p = 10$, respectively (in these cases a (s, S)-policy would do considerably better than a (s, Q)-policy). A major advantage of the (s, S, Q)-policy is that it includes both the (s, Q) and (s, S)-policy, so that it performs well in cases where a (s, Q)-policy is more suitable as well as in cases where a (s, S)-policy is more suitable.

Regarding the optimal values of s and Q, we have the following rules of thumb:

i) $\mu \uparrow \quad \Rightarrow \quad s^* \uparrow,\ Q^* \uparrow;$

ii) $K \uparrow \quad \Rightarrow \quad s^* \downarrow,\ Q^* \uparrow;$

iii) $p \uparrow \quad \Rightarrow \quad s^* \uparrow,\ Q^* \uparrow.$

However, there are exceptions, e.g., iii) is violated for $D = 0$, $L = 1$, $\mu = 10$, $K = 10$, or for $D = 1$, $L = 3$, $\mu = 10$, $K = 50$ (with Poisson demand). The search for the optimal (s, Q) pair or (s, S, Q) triple is complicated by the fact that $g(s, Q)$ and $g(s, S, Q)$ are not unimodal in s and Q, i.e., they may have two local minima. It turns out that at most two local minima exist and, if so, one lies in the region $\{(s, Q) : s < Q\}$ and the other in the region $\{(s, Q) : s \geq Q\}$. A good example is provided by the instance $D = 0$, $L = 1$, $\mu = 10$, $K = 10$, $p = 10$ and Poisson demand; Figure 5.5 shows a contour plot of the cost function $g(s, Q)$, i.e., the "iso-cost" lines $\{(s, Q) : g(s, Q) = nz\}$ for $n = 1, 2, \ldots$, from which it is clear that there are two local minima: one in (19,22) and the other in (20,14) (the global minimum). As a matter of fact, it is this phenomenon that leads to a bimodal optimal value function which in turn causes jumps and non-monotonicities in the optimal policy (see section 5.5).

Finally we note that, when varying the demand distribution ceteris paribus, the costs are lowest for Poisson demand, followed by two-point demand and geometric demand. Indeed, the geometric distribution has the highest coefficient of variation and the costs may be more than two times as high as for Poisson demand. Although the coefficient of variation of the two-point distribution is smaller than that of the Poisson distribution, it is not surprising that "all-or-nothing" type demand leads to higher costs.

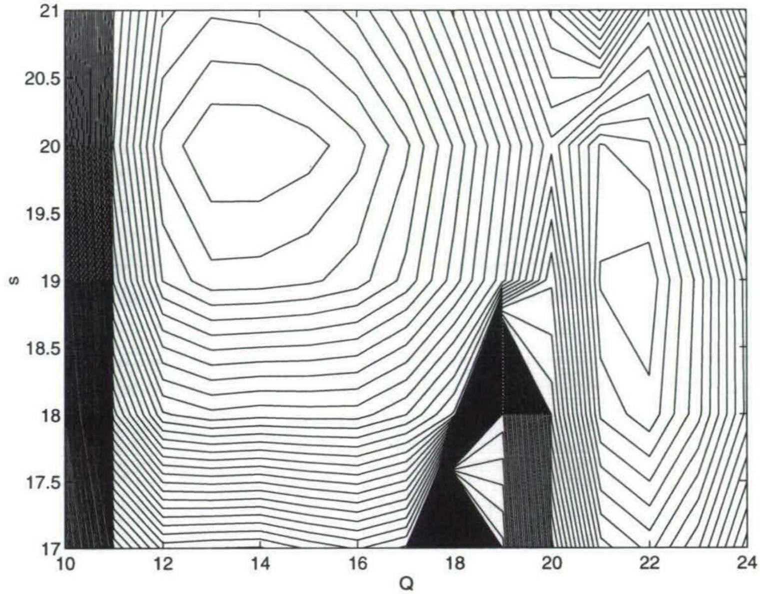

Figure 5.5: Contour plot of $g(s, Q)$ ($D=0, L=1, \mu=10, K=10, p=10$, Poisson)

Appendix 5.A: Poisson-stopped-sum distributions

If N has a Poisson distribution and $\{X_n;\ n = 1, 2, \ldots\}$ is a sequence of i.i.d. discrete random variables, then $Y := \sum_{n=1}^{N} X_n$ has a Poisson-stopped-sum or generalized Poisson distribution (see e.g. [Johnson&Kotz 1969], Chapter 9). Hence if the demand process is described by a compound Poisson process $\{Y(t)\}$, with $Y(t) = \sum_{n=1}^{N(t)} X_n$ (as for models C and CB), then the distribution of $Y(t)$ (and in particular lead-time demand $Y(L)$) falls into the class of Poisson-stopped-sum distributions. Here we discuss the following two special cases:

- if X_n has a Poisson distribution, then Y has a Neyman type A distribution;
- if X_n has a geometric distribution, then Y has a shifted Pólya-Aeppli distribution.

Neyman type A distribution

The Neyman type A distribution is defined as the Poisson-stopped-sum-Poisson distribution, i.e., the distribution of a Poisson number of Poisson random variables (see e.g. [Patil&Joshi 1968], section 47; [Johnson&Kotz 1969], section 9.6). If N has a Poisson(λ) distribution and the X_n are i.i.d. with a Poisson(ϕ) distribution, then $Y := \sum_{n=1}^{N} X_n$ has a Neyman type A distribution with parameters λ and ϕ. It is easily verified that the pmf is given by

$$p_k := \Pr\{Y = k\} = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} e^{-n\gamma} \frac{(n\gamma)^k}{k!} = e^{-\lambda} \frac{\gamma^k}{k!} \sum_{n=0}^{\infty} \frac{n^k (\lambda e^{-\gamma})^n}{n!} \quad (k = 0, 1, \ldots), \quad (5.111)$$

and that the probability generating function (pgf) is given by

$$G_Y(z) := E\{z^Y\} = e^{-\lambda(1-e^{-\phi(1-z)})} \quad (0 \le z \le 1). \quad (5.112)$$

The probabilities can be computed recursively using

$$\begin{aligned} p_0 &= e^{-\lambda(1-e^{-\phi})}; \\ p_k &= \frac{\lambda \phi e^{\phi}}{k} \sum_{j=0}^{k-1} \frac{\phi^j}{j!} p_{k-1-j} \quad (k = 1, 2, \ldots) \end{aligned} \quad (5.113)$$

(see also (5.23)). The solution of (5.113) and a useful alternative representation of the pmf is given by

$$p_k = e^{-\lambda(1-e^{-\phi})} \sum_{j=1}^{k} \sigma_k^{(j)} \lambda^j e^{-j\phi} \quad (k = 0, 1, \ldots), \quad (5.114)$$

with $\sigma_k^{(j)}$ the Stirling numbers of the second kind; see (4.30). Formula (5.114) only involves a polynomial of degree k, whereas (5.111) involves an infinite series.

Shifted Pólya-Aeppli distribution

If N has a Poisson(λ) distribution and the X_n are i.i.d. having a shifted (i.e., no mass in zero) geometric distribution with parameter r then $Y := \sum_{n=1}^{N} X_n$ has a so-called Pólya-Aeppli distribution with parameters λ and r (see e.g. [Patil&Joshi 1968], section 55; [Johnson&Kotz 1969], section 9.7). However, we are concerned with the case where the X_n have a geometric distribution (with mass in zero), and we refer to the resulting distribution of Y as the "shifted Pólya-Aeppli distribution". Since $\sum_{i=1}^{n} X_i$ has a negative binomial distribution, the pmf of Y is given by

$$p_0 = e^{-\lambda} + \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} r^n = e^{-\lambda(1-r)} \tag{5.115}$$

and

$$\begin{aligned} p_k &= \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \binom{n+k-1}{k-1} r^n (1-r)^k \\ &= (1-r)^k e^{-\lambda} \sum_{n=1}^{\infty} \frac{(k)_n}{(1)_n} \frac{(\lambda r)^n}{n!} \\ &= (1-r)^k e^{-\lambda} \Big(M(k,1,\lambda r) - 1 \Big) \quad (k = 1, 2, \ldots), \end{aligned} \tag{5.116}$$

where $M(a, b, z)$ is Kummer's function defined by (3.62) and (3.63). The recursive relation (5.23) becomes

$$\begin{aligned} p_0 &= e^{-\lambda(1-r)}; \\ p_k &= \frac{\lambda r}{k} \sum_{j=1}^{k} j(1-r)^j p_{k-j} \quad (k = 1, 2, \ldots). \end{aligned} \tag{5.117}$$

Chapter 6

Models allowing for production to order ($D > L$)

6.1 Introduction

The case $D > L$ in the general framework of section 4.2 is probably the most obvious extension of the service model to the production/inventory setting. It can be seen as the service model with an adjusted delay-limit of D–L and the additional possibility to include not only all waiting demand in a batch service but also future demand, incurring holding costs for demand that is serviceable before arrival. As explained in Chapter 4, both in the service model and the production/inventory model with $D > L$ the service activity can be started after a demand arrival and a queue of waiting demand may build up; this is what we mean by production to order. Whereas in the service model the only decision is when to do a batch service, the decision in the production/inventory model is two-fold: when to start a new production run and how many (additional) items to produce.

We consider the following two models:

PU: Periodic review, uncapacitated ($N = M = \infty$);

PC: Periodic review, one machine ($N = 1$, $M = \infty$).

As opposed to the previous chapter ($D \leq L$), the analysis for the case $N = \infty$ (model PU) turns out to be more straightforward than for the case $N = 1$ (model PC). Moreover, model PU is more closely linked to the discrete-time service model (which also assumes infinite capacity). Therefore we start with the uncapacitated model PU, and later show how the analysis is extended to the single-machine model PC. Note that according to Theorem 4.2 the assumption $N = 1$ is automatically satisfied if $D \geq 2L$, so that only the case $L < D < 2L$ for model PC requires a separate treatment. The assumption $M = \infty$ guarantees that a production batch will always include all waiting demand, and hence the relevant decision variable is how many *additional* items to produce in anticipation of future demand. Producing an extra item to inventory decreases the production or penalty costs but increases inventory holding costs; the problem is to find an optimal trade-off.

This chapter is organized as follows. In section 6.2 we formulate a SMDP to find an optimal production policy for the PU model. Next, in section 6.3, we extend the Critical-

Group policy and the (Extended) Total-Demand policy for the discrete-time service model to the Critical-Group-Production policy and the (Extended) Total-Demand-Production policy by adding a parameter for the excess production. In section 6.4 we present some numerical comparisons to study the performance of these heuristic policies. In section 6.5 we consider model PC and modify the SMDP of model PU. We complete the chapter in section 6.6 with some reflections on the continuous-review case, especially in connection with the continuous-time service model of Chapter 3.

6.2 A semi-Markov decision process for $N = \infty$

To find an optimal policy for model PU we can use a semi-Markov decision process (SMDP) with an appropriately chosen state space. When on-hand inventory is zero and no production run is underway (situation I of section 4.1), a queue of waiting demand builds up, and to satisfy demand within the delay-limit a new production run must be started within D–L periods. Therefore demand arriving in situation I starts with a residual delay-limit of D–L periods (instead of D periods). When on-hand inventory is positive and no production run is underway (situation III of section 4.1), arriving demand is satisfied from inventory and, due to the absence of waiting costs and capacity restrictions, it is clearly suboptimal to start a new production run before the inventory is depleted (in fact, it is suboptimal to start a new run before there is waiting demand with a residual delay-limit of zero; see also Theorem 2.4(i)). However, it is important to make a clear distinction between two types of waiting demand:

(a) demand that will be backordered from one of the production batches "in the pipeline";

(b) demand that will not be backordered from any of the production batches "in the pipeline".

In the following we reserve the term "waiting demand" for category (b), since the demand in category (a) is effectively satisfied; it will be backordered within L periods.

Thus, we take as decision epochs the endpoints of periods at which there is a queue of waiting demand (i.e., category-(b) demand). This yields a state space

$$\Omega := \Big\{\mathbf{r} := (r_0, \ldots, r_{D-L-1}) \mid r_i \in I\!N,\ i = 0, \ldots D-L-1\Big\}, \tag{6.1}$$

with r_i the number of waiting demands with a residual delay-limit of i periods. The action space is given by

$$A := \{\bar{0}\} \bigcup \{a \mid a = 0, 1, \ldots\}, \tag{6.2}$$

where action $a = \bar{0}$ stands for "do nothing" and action $a = 0, 1, \ldots$ for "start a production run for $r_{0,D-L-1} + a$ items, i.e., start a batch that accommodates all waiting demand and includes a additional items in anticipation of future demand. Notice that the action space is state-independent: all waiting demand should be included in any case, and the only decision variable is the size of the "production surplus".

Let $p(\mathbf{r}, \mathbf{s}; a)$, $\tau(\mathbf{r}; a)$ and $c(\mathbf{r}; a)$ ($\mathbf{r}, \mathbf{s} \in \Omega$, $a \in A$) denote the (one-step) transition probabilities, the expected transition times and the expected transition costs, respectively. If the process is in state $\mathbf{r}$ and it is decided not to start a production run ($a = \bar{0}$), then the demand with a residual delay-limit of zero (r_0) is lost and the next state is $(r_1, \ldots, r_{D-L-1}, k)$ if demand in the next period is k:

$$\begin{aligned} p(\mathbf{r}, \mathbf{s}; \bar{0}) &= q_k \quad (\mathbf{r} \in \Omega,\ \mathbf{s} = (r_1, \ldots, r_{D-L-1}, k),\ k \geq 0); && (6.3) \\ \tau(\mathbf{r}; \bar{0}) &= 1 \quad (\mathbf{r} \in \Omega); && (6.4) \\ c(\mathbf{r}; \bar{0}) &= p r_0 \quad (\mathbf{r} \in \Omega). && (6.5) \end{aligned}$$

To determine the transition law for $a \geq 0$, we utilize the discrete renewal function N_a and the forward recurrence time γ_a (see (4.14) and (4.21)). According to Lemma 5.1(i) the production surplus a is depleted in period $N_{a-1}+1$, while the first demand arrival after on-hand inventory has dropped to zero occurs in period N_a+1 (these periods coincide if $X_{N_{a-1}+1} > 1$). Also observe that the shortage (undershoot) in period N_a+1 amounts to γ_a (> 0). Now suppose that the process is in state $\mathbf{r}$ and that a production run for $r_{0,D-L-1}+a$ ($a \geq 0$) items is started. Then the next a items demanded will be backordered from this production batch, and the next decision epoch is the end of the period in which the $(a+1)^{\text{th}}$ item demanded arrives and joins the queue with a residual delay-limit of $D-L-1$ periods. Thus, the transition time is N_a+1 periods and the next state is $(0, \ldots, 0, \gamma_a)$:

$$\begin{aligned} p(\mathbf{r}, \mathbf{s}; a) &= g_k^{(a)} \quad (\mathbf{r} \in \Omega,\ \mathbf{s} = (0, \ldots, 0, k),\ k \geq 1); && (6.6) \\ \tau(\mathbf{r}; a) &= M_a + 1 \quad (\mathbf{r} \in \Omega), && (6.7) \end{aligned}$$

where

$$g_k^{(i)} := \Pr\{\gamma_i = k\} = \frac{q_{i+k}}{1 - q_0} + \sum_{j=1}^{i} q_{i+k-j} \sum_{n=1}^{\infty} q_j^{(n)} \quad (k \geq 1) \tag{6.8}$$

is the pmf of the forward recurrence time γ_i (see Theorem 4.5(i)). The transition costs for $a \geq 0$ consist of production costs and holding costs (lost sales only occur when $r_0 > 0$ and $a = \bar{0}$). Holding costs are only incurred if $S_L < a$ ($N_a > L$), and using (4.18) and (5.30) we find that

$$c(\mathbf{r}; a) = K + c(r_{0,D-L-1} + a) + c_h(a) \quad (\mathbf{r} \in \Omega,\ a \geq 0), \tag{6.9}$$

with

$$c_h(a) = E\Big\{h \sum_{n=L+1}^{\infty} (a - S_n)^+\Big\} = h \sum_{n=L+1}^{\infty} \sum_{k=0}^{a-1} Q_k^{(n)} = h \sum_{k=0}^{a-1} \Big(M_k - \sum_{n=1}^{L} Q_k^{(n)}\Big). \tag{6.10}$$

Incidently, using

$$\begin{aligned} E\{X_i \mid S_n \leq a\} &= \sum_{k=0}^{\infty} k \Pr\{X_i = k \mid S_n \leq a\} \\ &= \sum_{k=1}^{a} \frac{k \Pr\{X_i = k, S_n \leq a\}}{\Pr\{S_n \leq a\}} \\ &= \frac{1}{Q_a^{(n)}} \sum_{k=1}^{a} k q_k Q_{a-k}^{(n-1)} \quad (i = 1, \ldots, n), && (6.11) \end{aligned}$$

it follows that

$$\begin{aligned} c_h(a) &= \sum_{n=L+1}^{\infty} \Pr\{N_a = n\} E\Big\{ \sum_{i=L+1}^{n} h(a - S_i) \mid N_a = n \Big\} \\ &= h \sum_{n=L+1}^{\infty} \Pr\{N_a = n\}\Big((n-L)a - \sum_{i=L+1}^{n} E\{S_i \mid S_n \le a, S_{n+1} > a\}\Big) \\ &= h \sum_{n=L+1}^{\infty} \Pr\{N_a = n\}\Big((n-L)a - \sum_{i=L+1}^{n} iE\{X_i \mid S_n \le a\}\Big) \\ &= h \sum_{n=L+1}^{\infty} \Pr\{N_a = n\} \left((n-L)a - \Big(\frac{n(n+1)}{2} + \frac{L(L+1)}{2}\Big)\frac{1}{Q_a^{(n)}} \sum_{k=1}^{a} kq_k Q_{a-k}^{(n-1)}\right), \quad (6.12) \end{aligned}$$

an expression that apparently reduces to (6.10).

Now let $v(\mathbf{r})$ ($\mathbf{r} \in \Omega$) be the relative values and g the expected average costs per period of the optimal policy, then plugging (6.3)–(6.7) and (6.9) into the optimality equations we obtain

$$v(\mathbf{r}) = \min\Big\{z(\mathbf{r};\bar{0}), \min_{a=0,1,\ldots} \{K + c(r_{0,D-L-1} + a) + z(a)\}\Big\} \quad (\mathbf{r} \in \Omega), \qquad (6.13)$$

with

$$z(\mathbf{r};\bar{0}) := pr_0 - g + \sum_{k=0}^{\infty} q_k v(r_1, \ldots, r_{D-L-1}, k) \quad (\mathbf{r} \in \Omega) \qquad (6.14)$$

and

$$z(a) := h \sum_{k=0}^{a-1}\Big(M_k - \sum_{n=1}^{L} Q_k^{(n)}\Big) - (M_a + 1)g + \sum_{k=1}^{\infty} g_k^{(a)} v(0, \ldots, 0, k) \quad (a \ge 0). \qquad (6.15)$$

We may immediately conclude from (6.13) that the optimal production surplus is state-independent, yielding the following theorem.

Theorem 6.1 *Let π^* be an average-cost optimal policy with $\pi^*(\mathbf{r})$ the optimal action in state $\mathbf{r}$. Then*

$$\pi^*(\mathbf{r}) \in \{\bar{0}, a^*\} \quad (\mathbf{r} \in \Omega), \qquad (6.16)$$

with $a^* := \operatorname{argmin}_a\{ca + z(a)\}$ and $z(a)$ given by (6.15).

Remarks. (i) The optimality equations, and hence the optimal policy, only depend on D and L via $D-L$ and L. In fact, the only term that does not solely depend on $D-L$ is the holding cost $c_h(a)$ (see (6.10)).

(ii) It can be shown that the structural properties of the optimal policy for the discrete-time service model (see section 2.6.1) carry over to model PU, provided that D is replaced with $D-L$, a_B with K, b_B with c and b_I with p. For example, if a production batch is started in state $\mathbf{r}$, then a production batch will also be started in any state $\mathbf{r}'$ with $\mathbf{r}' \ge \mathbf{r}'$ (see Theorem 2.4(ii)).

6.3 Restricted policies

In this section we discuss three heuristic, simply-structured policies for model PU. The timing of the production run is based on the state vector $\mathbf{r}$, and an additional parameter determines the size of the production surplus whenever a production run is started; notice that by Theorem 6.1 there is no sense in letting the surplus depend on $\mathbf{r}$. We consider the following three policies, all being straightforward extensions of the corresponding policies for the discrete-time service model ($\pi(\mathbf{r})$ denotes the action in state $\mathbf{r}$):

- Critical-Group-Production policy with parameters K_{CG} and Q:
$$\pi(\mathbf{r}) = \begin{cases} Q & \text{if } r_0 \geq K_{\text{CG}} \\ \bar{0} & \text{else} \end{cases} \quad (\mathbf{r} \in \Omega);$$
- Total-Demand-Production policy with parameters K_{TD} and Q:
$$\pi(\mathbf{r}) = \begin{cases} Q & \text{if } r_{0,D-L-1} \geq K_{\text{TD}} \\ \bar{0} & \text{else} \end{cases} \quad (\mathbf{r} \in \Omega);$$
- Extended Total-Demand-Production policy with parameters K_{TD}, K_{CG} and Q:
$$\pi(\mathbf{r}) = \begin{cases} Q & \text{if } r_{0,D-L-1} \geq K_{\text{TD}} \textit{ and } r_0 \geq K_{\text{CG}} \\ \bar{0} & \text{else} \end{cases} \quad (\mathbf{r} \in \Omega).$$

6.3.1 The Critical-Group-Production policy

Defining the critical group as the number of demands with a residual delay-limit of zero (r_0), the Critical-Group-Production (CGP) policy with parameters K_{CG} and Q prescribes to start a production run for $r_{0,D-L-1} + Q$ items at any decision epoch at which the size of the critical group is at least K_{CG}. Defining a cycle as the number of periods between the start of two consecutive production runs, we are again led to a regenerative process. It is easily seen that a cycle consists of two phases: in the first phase the production surplus is depleted, while the second phase is similar to a cycle under the Critical-Group policy with delay-limit $D-L$ instead of D (see section 2.3).

As argued in the previous section, the extra production of a items is depleted in period $N_a + 1$ with an undershoot of γ_a. Therefore the second phase starts at the end of period $N_a + 1$ with γ_a waiting demands with a residual delay-limit of $D-L-1$; if $\gamma_a \geq K_{\text{CG}}$ then the next batch will be started $D-L-1$ periods later, while if $\gamma_a < K_{\text{CG}}$ the CG-policy with delay-limit $D-L$ takes effect. Consequently, we find for the cycle length of the CGP-policy (omitting the dependence on K_{CG} and Q for ease of notation):

$$S_{\text{CGP}} = \begin{cases} N_Q + 1 + S_{\text{CG}} & \text{if } \gamma_Q < K_{\text{CG}}; \\ N_Q + D - L & \text{if } \gamma_Q \geq K_{\text{CG}}, \end{cases} \tag{6.17}$$

where S_{CG} is given by (2.9) and (2.10) with D replaced by $D-L$ and K by K_{CG}.

It follows from (6.17) and (2.12) that

$$\begin{aligned} E\{S_{\text{CGP}}\} &= M_Q + G^{(Q)}_{K_{\text{CG}}-1}\big(D - L + \frac{1}{1 - Q_{K_{\text{CG}}-1}}\big) + (1 - G^{(Q)}_{K_{\text{CG}}-1})(D - L) \\ &= M_Q + D - L + \frac{G^{(Q)}_{K_{\text{CG}}-1}}{1 - Q_{K_{\text{CG}}-1}}, \end{aligned} \tag{6.18}$$

with $G_k^{(i)} := \sum_{j=1}^k g_j^{(i)}$ the cdf of γ_i. Next we turn to the total costs incurred in a cycle, consisting of production costs, holding costs and penalty costs. In the first phase only holding costs are incurred, and they are given by (6.10). In the second phase only penalty costs are incurred, and lost sales under the CGP-policy correspond to individual services under the CG-policy. Hence

$$C_{\text{CGP}} = \begin{cases} K + c(Z_{\text{CG}} + Q) + h \sum\limits_{n=L+1}^{\infty} (Q - S_n)^+ + p(\gamma_Q + Y_{\text{CG}}) & \text{if } \gamma_Q < K_{\text{CG}}; \\ K + c(\gamma_Q + S_{N_Q+2, N_Q+D-L} + Q) + h \sum\limits_{n=L+1}^{\infty} (Q - S_n)^+ & \text{if } \gamma_Q \geq K_{\text{CG}}, \end{cases} \tag{6.19}$$

where Y_{CG} (Z_{CG}) is the number of individual services (number of customers included in the batch service) in a cycle for the CG-policy with D replaced by $D-L$ and K by K_{CG}. It follows from (6.19), (6.10) and (2.16) that

$$\begin{aligned} E\{C_{\text{CGP}}\} &= K + c\Big(G_{K_{\text{CG}}-1}^{(Q)} \frac{\sum\limits_{k=K_{\text{CG}}}^{\infty} kq_k}{1 - Q_{K_{\text{CG}}-1}} + \sum_{k=K_{\text{CG}}}^{\infty} kg_k^{(Q)} + (D-1)\mu + Q\Big) + \\ & \quad h \sum_{k=0}^{Q-1} \Big(M_k - \sum_{n=1}^{L} Q_k^{(n)}\Big) + p\Big(\sum_{k=1}^{K_{\text{CG}}-1} kg_k^{(Q)} + G_{K_{\text{CG}}-1}^{(Q)} \frac{\sum\limits_{k=0}^{K_{\text{CG}}-1} kq_k}{1 - Q_{K_{\text{CG}}-1}}\Big). \end{aligned} \tag{6.20}$$

Applying the Renewal Reward Theorem we have that the expected average costs per period for the Critical-Group Production policy with parameters K_{CG} and Q are equal to

$$g_{\text{CGP}}(K_{\text{CG}}, Q) = \frac{E\{C_{\text{CGP}}\}}{E\{S_{\text{CGP}}\}}, \tag{6.21}$$

with $E\{C_{\text{CGP}}\}$ and $E\{S_{\text{CGP}}\}$ given by (6.18) and (6.20), respectively.

6.3.2 The Total-Demand-Production policy

Just like the CG-policy, the Total-Demand policy (see section 2.4) can be extended to the Total-Demand-Production (TDP) policy. Under the TDP-policy with parameters K_{TD} and Q, a production run for $r_{0,D-L-1} + Q$ items is started at any decision epoch at which the total number of waiting demands ($r_{0,D-L-1}$) is at least K_{TD}, but not earlier than $D-L$ periods since the last batch. The first phase, in which the production surplus is depleted, is identical to the first phase of the CGP-policy. The second phase is identical to a cycle of the TD-policy with delay-limit $D-L$, except for the fact that the demand in the first period is the undershoot γ_Q, which has a different distribution. Therefore we define

$$\tilde{X}_n := \begin{cases} \gamma_Q & \text{if } n = 1; \\ X_n & \text{if } n > 1, \end{cases} \qquad \tilde{S}_n := \sum_{i=1}^{n} \tilde{X}_i, \quad \tilde{S}_{m,n} := \sum_{i=m}^{n} \tilde{X}_i, \tag{6.22}$$

and

$$\tilde{L}_n := \begin{cases} \tilde{S}_n & \text{if } n \leq D-L; \\ \tilde{S}_{n-D+L+1,n} & \text{if } n > D-L. \end{cases} \tag{6.23}$$

(cf. also (2.26)). Now the cycle length can be written as

$$S_{\mathrm{TDP}} = N_Q + \tilde{S}_{\mathrm{TD}}, \tag{6.24}$$

with

$$\tilde{S}_{\mathrm{TD}} := \min\{n = D\text{–}L, D\text{–}L\text{+}1, \dots : \tilde{L}_n \geq K_{\mathrm{TD}}\} \tag{6.25}$$

(cf. also (2.27)). Analogously,

$$C_{\mathrm{TDP}} = K + c(\tilde{L}_{\tilde{S}_{\mathrm{TD}}} + Q) + h \sum_{n=L+1}^{\infty} (Q - S_n)^+ + p\tilde{Y}_{\mathrm{TD}}, \tag{6.26}$$

with

$$\tilde{Y}_{\mathrm{TD}} := \begin{cases} 0 & \text{if } \tilde{S}_{\mathrm{TD}} = D - L; \\ \sum_{n=1}^{\tilde{S}_{\mathrm{TD}}-D+L} \tilde{X}_n & \text{if } \tilde{S}_{\mathrm{TD}} > D - L. \end{cases} \tag{6.27}$$

It is possible to derive an explicit expression for the cost function $g_{\mathrm{TD}}(K_{\mathrm{TD}}, Q)$ from (6.24) and (6.26), but the resulting expression is of little practical use. To compute $g_{\mathrm{TD}}(K_{\mathrm{TD}}, Q)$ numerically there are a number of options:

- the probabilistic method described in section 2.4;
- the brute-force method described in Appendix 2.B;
- an embedded Markov chain on decision epochs, induced by the SMDP of section 6.2;
- value-determination (see (1.13) and (5.46)) for the SMDP of section 6.2.

6.3.3 The Extended Total-Demand-Production policy

The Extended Total-Demand-Production (ETDP) policy bases the production decision not exclusively on the size of the critical group (like the CGP-policy) or on total demand (like the TDP-policy), but on both these quantities (see also section 2.5.2). This results in a three-parameter (K_{TD}, K_{CG} and Q) policy: start a production run for $r_{0,D-L-1} + Q$ items at any decision epoch at which the total number of waiting demands ($r_{0,D-L-1}$) is at least K_{TD} *and* the number of critical demands (r_0) is at least K_{CG}. The cycle length becomes

$$S_{\mathrm{ETDP}} = N_Q + \tilde{S}_{\mathrm{ETD}}, \tag{6.28}$$

with

$$\tilde{S}_{\mathrm{ETD}} := \min\{n = D\text{–}L, D\text{–}L\text{+}1, \dots : \tilde{L}_n \geq K_{\mathrm{TD}} \text{ and } \tilde{X}_{n-D+L+1} \geq K_{\mathrm{CG}}\}. \tag{6.29}$$

The cycle costs C_{ETDP} follow directly from (6.26) and (6.27) with $\tilde{S}_{\mathrm{TD}}$ replaced by $\tilde{S}_{\mathrm{ETD}}$. Moreover, the computational methods for the TDP-policy also apply to the ETDP-policy (except for the probabilistic method).

6.4 Numerical comparisons

In this section we present a numerical comparison of the various restricted policies of the previous section and the optimal policy, being the solution of (6.13). As the state-space dimension and hence the computation time increases exponentially with $D-L$, we limit ourselves to values of L and D with $D–L \leq 3$. We assume a Poisson(λ) demand distribution truncated at $k_{\max}$ (i.e., $q_k := 0$ for $k > k_{\max}$) and consider the following cases:

- $D-L = 1$, $\lambda = 10$, $k_{\max} = 50$ (Table 6.1);
- $D-L = 2$, $\lambda = 5$, $k_{\max} = 15$ (Table 6.2);
- $D-L = 3$, $\lambda = 3$, $k_{\max} = 5$ (Table 6.3).

For all three cases we take $K \in \{10, 50\}$, $c = 0$, $h = 1$ and $p \in \{5, 10\}$. Since for $D-L = 1$ the optimal policy coincides with the optimal CGP-, TDP- and ETDP-policy, we only give the optimal policy in Table 6.1. For $D-L = 2$ the optimal policy can be represented as $(K_0^*, K_1^*, \ldots; Q^*)$, corresponding to

$$\pi^*(r_0, r_1) = \begin{cases} \bar{0} & \text{if } r_0 < K_{r_1}^*; \\ Q^* & \text{if } r_0 \geq K_{r_1}^* \end{cases} \tag{6.30}$$

(see also (2.62)). We use the same shorthand notation for $(K_0^*, K_1^*, \ldots)$ as in Table 2.2: n^m denotes a string of m n's, n-m denotes the string $n, n-1, \ldots, m$ $(m < n)$ and the last number is K_∞^*. For $D-L = 3$ the optimal policy is omitted because of its complex structure.

For the set of numerical examples considered here, the CGP-policy and the ETDP-policy perform very well and clearly outperform the TDP-policy. However, it should be expected that for larger values of $D-L$ and λ the performance of the CGP-policy will decrease, while the performance of the TDP-policy will increase (the ETDP-policy will remain close to optimal). A major advantage of the CGP-policy over the other policies is that it does not suffer from the curse of the dimensionality and hence can also be evaluated for larger values of $D-L$.

Under a CGP-policy with $K_{\text{CG}} = 1$ no lost sales are incurred, so that it does not depend on the penalty cost p; this is the "Only-Batch-Production" (OBP) policy. For low values of K (the set-up cost) c.q. high values of p the OBP-policy will do well, and this is confirmed by the cases where $K = 10$ in Tables 6.2 and 6.3. On the other hand, for very low values of p the "Never-Batch-Production" (NBP) policy, i.e., do not produce at all, may be (nearly) optimal.

6.5 A semi-Markov decision process for $N = 1$

In this section we consider model PC, i.e., there can be at most one production run at a time ($N = 1$). This limits the decision epochs to endpoints of periods with waiting demand in which either no production run was underway or a production run was completed. We focus on the case $L < D < 2L$, since by Theorem 4.2 the restriction $N = 1$ is not binding if

D	L	λ	K	p	$g(\pi^*)$	π^*	D	L	λ	K	p	$g(\pi^*)$	π^*
1	0	10	10	5	6.7699	(2,12)	3	2	10	10	5	3.3325	(1,27)
				10	6.8669	(1,12)					10	3.3325	(1,27)
			50	5	21.6398	(5,26)				50	5	13.3551	(3,38)
				10	22.3289	(3,26)					10	13.5408	(2,38)
2	1	10	10	5	4.5540	(1,19)	4	3	10	10	5	2.6228	(1,36)
				10	4.5540	(1,19)					10	2.6228	(1,36)
			50	5	16.7323	(4,31)				50	5	11.0039	(3,45)
				10	17.0856	(2,32)					10	11.1181	(2,46)

Table 6.1: Optimal (CGP-) policy ($D - L = 1$)

D	L	λ	K	p	$g(\pi^*)$	π^*	$g_{\rm CGP}(K^*_{\rm CG}, Q^*)$		$g_{\rm TDP}(K^*_{\rm TD}, Q^*)$		$g_{\rm ETDP}(K^*_{\rm TD}, K^*_{\rm CG}, Q^*)$	
2	0	5	10	5	4.0008	$(2,1;6)$	4.0022	(1,6)	4.5440	(5,7)	4.0008	(2,1,6)
				10	4.0022	$(1;6)$	4.0022	(1,6)	4.7906	(5,7)	4.0022	(1,1,6)
			50	5	13.6153	$(6\text{-}3;16)$	13.6508	(3,16)	14.0221	(8,16)	13.6153	(6,3,16)
				10	14.0318	$(3,2;16)$	14.0341	(2,16)	14.8223	(6,17)	14.0318	(3,2,16)
3	1	5	10	5	3.0803	$(2,1;10)$	3.0808	(1,10)	3.4519	(5,10)	3.0803	(2,1,10)
				10	3.0808	$(1;10)$	3.0808	(1,10)	3.5915	(4,11)	3.0808	(1,1,10)
			50	5	11.5841	$(5\text{-}3;16)$	11.5933	(3,16)	12.0257	(8,16)	11.5841	(5,3,16)
				10	11.8791	$(3,2;16)$	11.8802	(2,16)	12.6068	(6,16)	11.8791	(3,2,16)
4	2	5	10	5	2.4952	$(2,1;14)$	2.4952	(1,14)	2.7725	(5,14)	2.4952	(2,1,14)
				10	2.4952	$(1;14)$	2.4952	(1,14)	2.8516	(4,14)	2.4952	(1,1,14)
			50	5	10.4272	$(5\text{-}3;18)$	10.4310	(3,17)	10.8177	(7,18)	10.4272	(5,3,18)
				10	10.6514	$(3,2;18)$	10.6518	(2,18)	11.2846	(6,18)	10.6514	(3,2,18)
5	3	5	10	5	2.1341	$(1;16)$	2.1341	(1,16)	2.3734	(5,16)	2.1341	(1,1,16)
				10	2.1341	$(1;16)$	2.1341	(1,16)	2.4269	(4,16)	2.1341	(1,1,16)
			50	5	9.6548	$(4\text{-}2;21)$	9.6633	(2,21)	9.9859	(7,22)	9.6548	(4,2,21)
				10	9.8240	$(2,1;22)$	9.8261	(1,22)	10.3570	(6,22)	9.8240	(2,1,22)

Table 6.2: Numerical comparison of optimal, CGP-, TDP- and ETDP-policy ($D - L = 2$)

D	L	λ	K	p	$g(\pi^*)$	$g_{\rm CGP}(K^*_{\rm CG}, Q^*)$		$g_{\rm TDP}(K^*_{\rm TD}, Q^*)$		$g_{\rm ETDP}(K^*_{\rm TD}, K^*_{\rm CG}, Q^*)$	
3	0	3	10	5	2.7728	2.7750	(1,5)	3.2934	(6,5)	2.7745	(2,1,5)
				10	2.7750	2.7750	(1,5)	3.5157	(5,5)	2.7750	(1,1,5)
			50	5	9.6675	9.7288	(3,11)	9.9784	(9,11)	9.6849	(7,3,11)
				10	10.0943	10.0970	(2,11)	10.8490	(7,12)	10.0947	(4,2,12)
4	1	3	10	5	2.2867	2.2870	(1,6)	2.7367	(5,6)	2.2867	(2,1,6)
				10	2.2870	2.2870	(1,6)	2.8864	(5,6)	2.2870	(1,1,6)
			50	5	9.2450	9.3120	(3,10)	9.6094	(8,11)	9.2747	(7,2,10)
				10	9.6322	9.6527	(2,10)	10.4204	(7,11)	9.6506	(4,1,10)
5	2	3	10	5	2.0884	2.0886	(1,6)	2.5123	(5,6)	2.0884	(2,1,6)
				10	2.0886	2.0886	(1,6)	2.6620	(5,6)	2.0886	(1,1,6)
			50	5	8.5507	8.6274	(2,11)	8.9179	(8,11)	8.5775	(6,2,11)
				10	8.8453	8.8691	(1,11)	9.6444	(6,11)	8.8630	(3,1,11)

Table 6.3: Numerical comparison of optimal, CGP-, TDP- and ETDP-policy ($D - L = 3$)

$D \geq 2L$ (any reasonable policy for $D \geq 2L$ will ensure that there is at most one production run at any time).

The transitions for $a = \bar{0}$ are clearly the same as for $N = \infty$ and thus given by (6.3)–(6.5). Suppose that the process is in state $\mathbf{r}$ and that a batch for $r_{0,D-L-1} + a$ ($a \geq 0$) items is started. Upon completion of the run at the end of period L, we have the following sequence of events:

1 the demand $r_{0,D-L-1}$ is backordered;

2 the lead-time demand S_L is backordered from the production surplus a on a first-come-first-served basis;

3a if $S_L < a$ then on-hand inventory becomes $a - S_L$, and the next decision epoch is N_a+1 periods after the previous one (when the inventory is depleted);

3b if $S_L \geq a$ then a queue of waiting demand remains, and the next decision epoch is L periods after the previous one.

Now, since the next run cannot be started before the batch is completed, demand that arrives in period $L-(D-L) = 2L-D$ or earlier can only be backordered from the current production batch; hence if $N_a < 2L-D$, the demand in periods $N_a+1, \ldots, 2L-D$ is lost. On the other hand, if $N_a \geq L$ (or $S_L \leq a$) then all lead-time demand can be satisfied from the production batch a and the remaining $a - S_L$ items become on-hand inventory. In the intermediate case $2L-D \leq N_a < L$ no demand is lost and a queue of waiting demand remains. Defining $\mathbf{R}(a)$ as the state at the next decision epoch when action a is taken (a random variable), we see that

$$\mathbf{R}(a) = \begin{cases} (X_{2L-D+1}, \ldots, X_L) & \text{if } N_a < 2L - D; \\ (0, \ldots, 0, \gamma_a, X_{N_a+2}, \ldots, X_L) & \text{if } 2L - D \leq N_a < L - 1; \\ (0, \ldots, 0, \gamma_a) & \text{if } N_a \geq L - 1. \end{cases} \tag{6.31}$$

If $S_{L-1} > a$, the next decision epoch is at the end of the production run and the state depends on the lead-time demand. Define $\mathbf{k} := (k_1, \ldots, k_L)$, and $T(\mathbf{k}; a)$ as the state at the end of the production run (after backordering) given that $\{X_1 = k_1, \ldots, X_L = k_L\}$ and $k_{1,L-1} > a$. Then

$$T(\mathbf{k}; a) = \begin{cases} (k_{2L-D+1}, \ldots, k_L) & \text{if } k_{1,2L-D} > a \\ (0, \ldots, 0, k_{1,m} - a, k_{m+1}, \ldots, k_L) & \text{if } k_{1,m-1} \leq a \text{ and } k_{1,m} > a \\ & (2L - D < m < L - 1), \end{cases} \tag{6.32}$$

so that

$$p(\mathbf{r}, T(\mathbf{k}; a); a) = q_{k_1} \cdots q_{k_L} \quad (k_{1,D-1} > a > 0). \tag{6.33}$$

If $S_{L-1} \leq a$, the next decision epoch is at the end of period N_a+1 ($\geq L$) and the state is $(0, \ldots, 0, k)$ with k the shortage. Consequently,

$$p(\mathbf{r}, (0, \ldots, 0, k); a) = \Pr\{N_a \geq L{-}1, \gamma_a = k\} = \sum_{l=0}^{a} q_l^{(L-1)} g_k^{(a-l)} \quad (\mathbf{r} \in \Omega, k > 0, a > 0), \tag{6.34}$$

Since the next decision epoch is the end of period $\max\{L, N_a+1\}$,

$$\begin{aligned}
\tau(\mathbf{r};a) &= L\sum_{n=1}^{L-1}\Pr\{N_a=n\}+\sum_{n=L}^{\infty}(n+1)\Pr\{N_a=n\} \\
&= L+\sum_{n=L}^{\infty}(n+1-L)\Pr\{N_a=n\}=L+\sum_{n=L}^{\infty}\Pr\{N_a\ge n\} \\
&= L+M_a-\sum_{n=1}^{L-1}\Pr\{S_n\le a\}=L+M_a-\sum_{n=1}^{L-1}Q_a^{(n)} \quad (a>0). \qquad (6.35)
\end{aligned}$$

The expected transition costs consist of production costs $K+c(r_{0,D-L-1}+a)$, holding costs $c_h(a)$ and penalty costs $c_p(a)$, i.e.,

$$c(\mathbf{r};a)=K+c(r_{0,D-L-1}+a)+c_h(a)+c_p(a) \quad (\mathbf{r}\in\Omega,\ a>0), \qquad (6.36)$$

where $c_h(a)$ is given by (6.10). Penalty costs are only incurred if $S_{2L-D}>a$ ($N_a<2L-D$):

$$c_p(a)=E\{(S_{2L-D}-a)^+\}=p\Big((2L-D)\mu-\sum_{k=1}^{a}\bar{Q}_k^{(2L-D)}\Big) \qquad (6.37)$$

(cf. also (5.31)).

Substitution of (6.33)–(6.37) and (6.10) into the optimality equations yields

$$v(\mathbf{r})=\min\Big\{z(\mathbf{r};\bar{0}),\ \min_{a=0,1,\ldots}\{K+c(r_{0,D-L-1}+a)+z(a)\}\Big\} \quad (\mathbf{r}\in\Omega), \qquad (6.38)$$

with

$$z(\mathbf{r};\bar{0})=pr_0-g+\sum_{k=0}^{\infty}q_k v(r_1,\ldots,r_{D-L-1},k) \quad (\mathbf{r}\in\Omega) \qquad (6.39)$$

and

$$\begin{aligned}
z(a) &= h\sum_{k=0}^{a-1}\Big(M_k-\sum_{n=1}^{L}Q_k^{(n)}\Big)+p\Big((2L-D)\mu-\sum_{k=1}^{a}\bar{Q}_k^{(2L-D)}\Big)-\Big(L+M_a-\sum_{n=1}^{L-1}Q_a^{(n)}\Big)g+ \\
&\quad \sum_{l=0}^{a}q_l^{(L-1)}\gamma_k^{(a-l)}v(0,\ldots,0,k)+\sum_{\mathbf{k}:k_{1,L-1}>a}q_{k_1}\cdots q_{k_L}v(T(\mathbf{k};a)) \quad (a\ge 0). \qquad (6.40)
\end{aligned}$$

6.6 A continuous-review model

We close this chapter with some notes on a continuous-review version of model PU, which can be seen as the extension of the continuous-time service model of Chapter 3 to the production/inventory setting. The model assumptions are:

- A Poisson demand process $\{N(t)\}$ with rate λ;
- A constant delay-limit of D time units;
- A constant production lead time of L time units, with $L < D$;
- No capacity restrictions, i.e., $N = M = \infty$;

For this model the policies of Chapter 3 can be applied with an additional parameter for the size of the production surplus. To illustrate, we consider the extension of the continuous-time TD-policy; other policies (such as the GCG-policy) are extended analogously.

Under the continuous-review Total-Demand-Production (TDP) policy, a production run for $K_{\text{TD}} + Q$ items is started whenever the number of waiting demands reaches the level K_{TD} (note that there is no overshoot). Let $X_{\text{TDP}}(t)$ be the number of waiting demands at time t under the TDP-policy and $B_i := \sum_{j=1}^{i} A_j$ the n^{th} demand arrival epoch. It is easily verified that the epochs at which a production run is started are regeneration epochs for $\{X_{\text{TDP}}(t)\}$. The first Q demands in any cycle are satisfied from the production surplus, after which the time until the next production run is equivalent to a cycle of the TD-policy with D replaced by $D-L$ and K by K_{TD}. Using (3.14) it follows that

$$E\{S_{\text{TDP}}\} = E\{B_Q + T_{K_{\text{TD}}}^{(D-L)}\} = \frac{Q}{\lambda} + E\{T_{K_{\text{TD}}}^{(D-L)}\} \tag{6.41}$$

(recall that $T_K^{(C)}$ is the first entrance time into level K of a M/D/∞ queue with constant service times C). The expected costs in a cycle consist of production and holding costs (that only depend on Q), and penalty costs (that only depend on K_{TD}):

$$E\{C_{\text{TDP}}\} = K + c(K_{\text{TD}} + Q) + c_h(Q) + c_p(K_{\text{TD}}). \tag{6.42}$$

To compute the expected holding costs $c_h(Q)$, suppose that a cycle starts at time 0. Then the production batch is completed at time L, and hence holding costs are only incurred if the production surplus is not yet depleted at time L, i.e., if $N(L) < Q$. Conditioning on $N(L)$ we find that

$$\begin{aligned} c_h(Q) &= h\sum_{i=0}^{Q} e^{-\lambda L}\frac{(\lambda L)^i}{i!}\sum_{j=1}^{Q-i}\frac{j}{\lambda} \\ &= \frac{h}{2\lambda}\sum_{i=0}^{Q}(Q-i)(Q-i+1)e^{-\lambda L}\frac{(\lambda L)^i}{i!} \\ &= \frac{h}{2\lambda}\Big(Q(Q+1)Q_{Q-1}(L) - 2Q\,\lambda L Q_{Q-2}(L) + (\lambda L)^2 Q_{Q-3}(L)\Big), \end{aligned} \tag{6.43}$$

with

$$Q_k(t) := \Pr\{N(t) \le k\} = \begin{cases} 0 & \text{if } k < 0; \\ \displaystyle\sum_{i=0}^{k} e^{-\lambda t}\frac{(\lambda t)^i}{i!} & \text{if } k \ge 0. \end{cases} \tag{6.44}$$

Since lost sales for the TDP-policy correspond to individual services for the TD-policy, it follows from (3.15) and (3.22) that

$$c_p(K_{\mathrm{TD}}) = pE\{N_{K_{\mathrm{TD}}}^{(D-L)} - K_{\mathrm{TD}}\} = p\lambda E\{T_{K_{\mathrm{TD}}}^{(D-L)}\} - K_{\mathrm{TD}}. \tag{6.45}$$

Combining (6.41), (6.42) and (6.45) we conclude that

$$g_{\mathrm{TDP}}(K_{\mathrm{TD}}, Q) = \frac{K + c(K_{\mathrm{TD}} + Q) + c_h(Q) + p\big(\lambda E\{T_{K_{\mathrm{TD}}}^{(D-L)}\} - K_{\mathrm{TD}}\big)}{\frac{Q}{\lambda} + E\{T_{K_{\mathrm{TD}}}^{(D-L)}\}}, \tag{6.46}$$

with $c_h(Q)$ given by (6.43). Minimization of (6.46) with respect to K_{TD} and Q is facilitated by the fact that no terms depend on K_{TD} and Q simultaneously, i.e., both the numerator and the denominator are separable in K_{TD} and Q.

Chapter 7

Capacitated models

7.1 Introduction

All production/inventory models considered thus far hinge upon the assumption that there is no prespecified upper bound on the size of a production batch (i.e., $M = \infty$). Moreover, the number of machines or the maximum number of simultaneous production runs (N) was assumed to be either 1 or infinity. In some cases the assumption $N = 1$ is not restrictive, in the sense that a policy for a model with $N > 1$ does not improve upon the same policy for the same model with $N = 1$. Two examples are:

- A (s, Q) or (s, S, Q)-policy with $Q > s$ (see section 4.3.1);
- A CGP-policy or an optimal policy for $D \geq 2L$ and $M = \infty$ (see Theorem 4.2).

In this chapter we return to the general framework of section 4.2 and consider the capacitated model with N identical machines and a production capacity of M items on each machine. Notice that the total production capacity per unit of time is given by $\frac{NM}{L}$. Hence if $\frac{NM}{L} >> \mu$, with μ the average demand per unit of time, the capacity constraints will have little influence on the optimal policy. On the other hand, if $\frac{NM}{L} << \mu$ then the optimal policy will likely be to fully utilize the available capacity. In general the capacity constraints may have a severe influence on the optimal policy, and then it is important to make efficient use of the available capacity.

In the next section we will focus on a general periodic-review model where the delay-limit and the production lead time equal D and L periods, respectively. A complete state description for this model requires D–1 variables for the residual delays (like in the models of Chapter 2 and Chapter 6), as well as $L-1$ variables for the outstanding production batches (like in the lost-sales inventory model with an order lead time of L periods; see subsection 4.3.1). This allows us to formulate a MDP with a $(D+L-1)$-dimensional state space, where the cases $D = 0$ and $L = 0$ must be treated separately (subsections 7.2.1–7.2.4). Clearly the MDP is only of practical interest for small values of $D+L$, and the optimal production policies may be very complex. In subsection 7.2.5 we compute the optimal policy for some simple cases, thereby illustrating the influence of the capacity constraints. A major problem, that we have already encountered in Chapter 2, is that the analysis of almost any dynamic (state-dependent) heuristic policy requires the same multi-dimensional state description as the MDP for the optimal policy. Therefore we resort to an appealing static policy in section 7.3, namely the (T, Q)-policy: start a new production run for Q items every T periods. The costs for this policy can be computed by means of a Markov chain analysis, where the capacity constraints are incorporated through a lower bound on T and an upper bound on Q. In spite of its simplicity the (T, Q)-policy performs remarkably well in a lot of cases.

Finally, in section 7.4, we discuss a model extension regarding switch-over times, which can be applied to the planning of trucks transporting cargo from a depot to a wholesaler.

7.2 A general periodic-review model

In this section we consider the general periodic-review capacitated production/inventory model, characterized by

- a delay-limit of D periods;
- a production lead time of L periods;
- N machines;
- a production capacity of M for each machine

(see also section 4.2). The cost structure is given by (4.2), and as usual we use the criterion of expected average costs per unit of time. We assume w.l.o.g. that $c = 0$, since the optimal policy for a model with parameters c and p is the same as for a model with parameters $c' = 0$ and $p' = p-c$, ceteris paribus (this is not true for the criterion of expected discounted costs).

In an optimal production policy for this model the production decisions will depend on on-hand inventory, the size and residual production time of all outstanding batches and the residual delay-limit of all waiting demands. For a periodic-review model, outstanding batches with the same residual production time as well as waiting demands with the same residual delay-limit can be grouped together into $L-1$ and $D-1$ groups, respectively. Therefore it is possible to construct a MDP with a $(D+L-1)$-dimensional state space. In the next subsections we present this MDP-formulation, not only for the case $D > 0$ and $L > 0$ but also for the special cases $D = 0$ and/or $L = 0$. Due to the curse of dimensionality, the optimal production policy can only be computed numerically for small values of $D+L$. Although the optimal policy is generally too complex to be of any practical use, it is interesting to compare the minimum cost with the expected costs of various heuristic policies. Moreover, studying the optimal policy may lead to insight as to what kind of heuristic policies are likely to perform well. In subsection 7.2.5 we compute the optimal policy for combinations of L and D with $D+L \leq 1$. This will illustrate the impact of capacity constraints, as well as the versatility of the general periodic-review model.

7.2.1 The case $D > 0$ and $L > 0$

We specify all ingredients for the MDP in turn:

- The state description $\mathbf{z}$;
- The state space Ω;
- The action spaces $A(\mathbf{z})$ for $\mathbf{z} \in \Omega$;
- The one-step transition probabilities $p(\mathbf{z}, \mathbf{z}'; a)$ for $\mathbf{z}, \mathbf{z}' \in \Omega$ and $a \in A(\mathbf{z})$;
- The one-step transition costs $c(\mathbf{z}; a)$ for $\mathbf{z} \in \Omega$ and $a \in A(\mathbf{z})$.

A complete state description if both $L > 0$ and $D > 0$ is given by

$$\mathbf{z} := (i; j_1, \ldots, j_{L-1}; r_1, \ldots, r_{D-1}), \tag{7.1}$$

with

$$\begin{aligned} i &:= \text{on-hand inventory at the start of a period} \\ j_n &:= \text{number of items that will be completed in } n \text{ periods } (n = 1, \ldots, L-1) \\ r_n &:= \text{number of demands with a residual delay-limit of } n \text{ periods } (n = 1, \ldots, D-1). \end{aligned}$$

Note that under an optimal policy $i > 0$ implies that $r_1 = \cdots = r_{D-1} = 0$, because there is no sense in letting customers wait if items are available. Moreover, according to the capacity constraints, the number of outstanding batches (machines in use) cannot exceed N and the size of a production batch cannot exceed M. Obviously, if $M = \infty$ then at any decision epoch at most one new machine will be started, implying that at most L batches are outstanding at any time. However, if $M < \infty$ then it may be desirable to produce more than M items and to start more than one machine at the same decision epoch (incurring a set-up cost for each additional machine). Since the number of machines needed to produce j items is $\lceil \frac{j}{M} \rceil$, the number of machines in use follows directly from the state vector $\mathbf{z}$. Specifically, define

$$N(\mathbf{z}) \;:=\; \text{number of machines in use in state } \mathbf{z},$$

then

$$N(\mathbf{z}) = \sum_{n=1}^{L-1} \left\lceil \frac{j_n}{M} \right\rceil. \tag{7.2}$$

Since the number of machines in use cannot exceed N, the state space is given by

$$\Omega := \left\{ \mathbf{z} : i = 0,\ N(\mathbf{z}) \le N \right\} \bigcup \left\{ \mathbf{z} : r_1 = \cdots = r_{D-1} = 0,\ N(\mathbf{z}) \le N \right\} \tag{7.3}$$

and the action spaces are given by

$$A(\mathbf{z}) := \left\{ 0, \ldots, (N - N(\mathbf{z}))\, M \right\} \quad (\mathbf{z} \in \Omega) \tag{7.4}$$

(where an action denotes the number of items to be produced when the system is in state $\mathbf{z}$). The state transitions depend on the current state $\mathbf{z}$, the size of the new production batch j_L (decision variable) and the number of demands in the review period r_D (realization). By defining the *transfer function*

$$T(\mathbf{z};j_L;r_D) \;:=\; \begin{array}{l}\text{state at the next decision epoch when the current state is } \mathbf{z},\\ \text{a production batch for } j_L \text{ items is started and } r_D \text{ demands}\\ \text{arrive during the period} \quad (\mathbf{z}\in\Omega,\ j_L\in A(\mathbf{z}),\ r_D=0,1,\ldots),\end{array}$$

we have that

$$p(\mathbf{z},T(\mathbf{z};j_L;r_D);j_L)=q_{r_D}, \tag{7.5}$$

and it remains to specify $T(\mathbf{z};j_L;r_D)$. In doing this, we use the partial sums $r_{m,n}:=\sum_{l=m}^{n} r_l$ and distinguish between three subsets of states.

I $\{\mathbf{z}: i>0\}$.
When on-hand inventory is positive, it will remain positive if $r_D\le i+j_1$ and else a queue will start to build up:

$$T(\mathbf{z};j_L;r_D)=\begin{cases}(i-r_D+j_1;j_2,\ldots,j_L;0,\ldots,0) & \text{if } r_D\le i+j_1;\\ (0;j_2,\ldots,j_L;0,\ldots,0,r_D-i-j_1) & \text{if } r_D>i+j_1.\end{cases} \tag{7.6}$$

II $\{\mathbf{z}: i=0,\ j_1>0\}$.
When on-hand inventory is zero but a batch is completed in one period, we need to specify which of the waiting (and possibly arriving) demands can be satisfied from the incoming batch. To do so, we use deterministic analogues of the discrete renewal function and the forward recurrence time,

$$n_j:=\max\{n\le D: r_{1,n}\le j\},\quad g_j:=r_{1,n_j+1}-j \quad (j=0,1,\ldots) \tag{7.7}$$

(note that $n_j:=0$ if $r_1>j$). Using this notation, we see that a queue remains if $n_{j_1}<D$, while all waiting and arriving demand is cleared and on-hand inventory becomes positive if $n_{j_1}\ge D$:

$$T(\mathbf{z};j_L;r_D)=\begin{cases}(0;j_2,\ldots,j_L;0,\ldots,0,g_{j_1},r_{n_{j_1}+2},\ldots,r_D) & \text{if } n_{j_1}<D-1;\\ (0;j_2,\ldots,j_L;0,\ldots,0,g_{j_1}) & \text{if } n_{j_1}=D-1;\\ (j_1-r_{1,D};j_2,\ldots,j_L;0,\ldots,0) & \text{if } n_{j_1}>D-1.\end{cases} \tag{7.8}$$

III $\{\mathbf{z}: i=0,\ j_1=0\}$.
In these states no demand can be satisfied, r_1 demands are lost, and r_D demands join the queue of waiting demand:

$$T(\mathbf{z};j_L;r_D)=(0;j_2,\ldots,j_L;r_2,\ldots,r_D). \tag{7.9}$$

It is easily verified that (7.6), (7.8) and (7.9) can be summarized through a single formula,

$$T(\mathbf{z};j_L;r_D)=\big((i+j_1-r_{1,D})^+;j_2,\ldots,j_L;r_2-(i+j_1-r_1)^+,\ldots,r_D-(i+j_1-r_{1,D-1})^+\big). \tag{7.10}$$

As for the transition costs, they consist of set-up costs, penalty costs and holding costs. If action j_L is taken, then $\lceil \frac{j_L}{M} \rceil$ machines need to be started and set-up costs of $K \lceil \frac{j_L}{M} \rceil$ are incurred. Lost sales occur when $r_1 > 0$ (implying that $i = 0$) and $j_1 < r_1$, resulting in penalty costs of $p(r_1 - j_1)^+$. Holding costs are incurred if $i > 0$ and $r_D < i$, and they amount to $h \sum_{r_D=0}^{i-1} q_{r_D}(i - r_D)$. Consequently, the transition costs are given by

$$c(\mathbf{z}; j_L) = K\left\lceil \frac{j_L}{M} \right\rceil + p(r_1 - j_1)^+ + h \sum_{r_D=0}^{i-1} q_{r_D}(i - r_D) \quad (\mathbf{z} \in \Omega,\ j_L \in A(\mathbf{z})). \tag{7.11}$$

Using (7.5) and (7.11) the optimality equations can be written as

$$v(\mathbf{z}) = \min_{j_L \in A(\mathbf{z})} \left\{ K\left\lceil \frac{j_L}{M} \right\rceil + p(r_1 - j_1)^+ - g + \sum_{r_D=0}^{\infty} q_{r_D}\Big(h(i - r_D)^+ + v(T(\mathbf{z}; j_L; r_D))\Big) \right\}, \tag{7.12}$$

with $A(\mathbf{z})$ and $T(\mathbf{z}; j_L; r_D)$ given by (7.4) and (7.10), respectively.

7.2.2 The case $D = 0$ and $L > 0$

The case $D = 0$ corresponds to a periodic-review lost-sales inventory model with an order lead time of L periods, at most N outstanding orders, and a maximal order size of M items. In section 4.3 we have already presented the optimality equations for the case $D = 0$, $N = \infty$ and $M = \infty$ (see (4.5)). These optimality equations are easily generalized to the case $N < \infty$ and $M < \infty$, by using the restricted state space

$$\Omega := \Big\{ \mathbf{z} : N(\mathbf{z}) \leq N \Big\} \tag{7.13}$$

and the restricted action spaces (7.4), and by setting $c(a) = K\lceil \frac{a}{M} \rceil$.

Therefore, using the notation of the previous section, we have that

$$v(\mathbf{z}) = \min_{j_L \in A(\mathbf{z})} \left\{ K\left\lceil \frac{j_L}{M} \right\rceil - g + \sum_{k=0}^{\infty} q_k \Big(h(i-k)^+ + p(k-i)^+ + v(T(\mathbf{z}; j_L; k)\Big) \right. \tag{7.14}$$

with

$$\mathbf{z} := (i; j_1, \ldots, j_{L-1}) \tag{7.15}$$

and

$$T(\mathbf{z}; j_L; k) = \Big((i-k)^+ + j_1; j_2, \ldots, j_L\Big). \tag{7.16}$$

7.2.3 The case $D > 0$ and $L = 0$

For $L = 0$ the state description is given by

$$\mathbf{z} := (i; r_0, \ldots, r_{D-1}). \tag{7.17}$$

Note that an additional state variable r_0 for the number of demands with a residual delay-limit of 0 is needed, because these can be satisfied by a new – instantaneously available – production batch.

Since there are no outstanding batches, the state space reduces to

$$\Omega := \{\mathbf{z} : i = 0\} \bigcup \{\mathbf{z} : r_0 = \cdots = r_{D-1} = 0\}, \tag{7.18}$$

and the action spaces reduce to

$$A(\mathbf{z}) := \{0, \ldots, NM\} \quad (\mathbf{z} \in \Omega). \tag{7.19}$$

It is easily verified that the optimality equations are now given by

$$v(\mathbf{z}) = \min_{j_0=0,\ldots,NM} \Big\{ K\Big\lceil \frac{j_0}{M} \Big\rceil + p(r_0 - j_0)^+ + \sum_{r_D=0}^{\infty} q_{r_D} \Big(h(i + j_0 - r_{0,D})^+ + v(T(\mathbf{z}; j_0; r_D)) \Big) \Big\}, \tag{7.20}$$

with

$$T(\mathbf{z}; j_0; r_D) = \Big((i + j_0 - r_{0,D})^+; r_1 - (i + j_0 - r_0)^+, \ldots, r_D - (i + j_0 - r_{0,D-1})^+ \Big). \tag{7.21}$$

Note that this case is equivalent to the case of a production capacity of NM items and a discontinuous production cost function

$$c(i) = K\Big\lceil \frac{i}{M} \Big\rceil \quad (i = 0, 1, \ldots, NM). \tag{7.22}$$

Remarks. (i) Since either $i = 0$ or $r_0 = 0$, the two state variables i and r_0 can be combined into a single variable $i' := i - r_0$. For $D = 1$, i' is conveniently interpreted as "net inventory" (inventory on hand minus backorders).

(ii) The case $D > 0, L = 0$ also includes the discrete-time service model of Chapter 2 by setting $N = 1$, $M = \infty$ and $A(\mathbf{z}) = \{0, r_{0,D-1}\}$ (either do nothing or clear all waiting demand). Since it is impossible to build up inventory the state vector then reduces to $\mathbf{z} = (r_0, \ldots, r_{D-1})$ (cf. also (2.2)).

7.2.4 The case $D = 0$ and $L = 0$

This case corresponds to a periodic-review lost-sales inventory model with instantaneous deliveries and a discontinuous ordering cost function as in (7.22). The optimality equations are given by

$$v(i) = \min_{j=0,\ldots,NM} \Big\{ K\Big\lceil \frac{j}{M} \Big\rceil + l(i + j) + \sum_{k=0}^{i+j-1} q_k v(i + j - k) + \bar{Q}_{i+j} v(0) \Big\} \quad (i = 0, 1, \ldots), \tag{7.23}$$

with

$$l(i) := \sum_{k=0}^{\infty} q_k \Big(h(i - k)^+ + p(k - i)^+ \Big) \tag{7.24}$$

the one-period loss function (cf. also (5.7)). By letting $M \to \infty$ in (7.23) we obtain the standard optimality equations for a fixed ordering cost, for which it is well-known that the optimal policy is of the (s, S)-type. For $M < \infty$ (multiple set-ups) the (s, S)-structure is lost; see the next subsection for numerical examples.

7.2.5 Numerical results

Clearly, the presence of capacity constraints (i.e., $N < \infty$ and/or $M < \infty$) may have a considerable influence on the form of the optimal policy as well as the minimum cost. For example, if $N > 1$ and $M < \infty$ then the production quantity is likely to be a multiple of M, since this minimizes the per-item production cost (see also (7.22)). We now illustrate this by considering some simple cases of the general model of the previous section.

In Tables 7.1–7.3 we present the minimum cost and the optimal policy for the cases $D = L = 0$, $D = 0, L = 1$, and $D = 1, L = 0$, respectively, for $N \in \{1,2,3\}$, $M \in \{5,10,15\}$, X_n Poisson(10) distributed, $K \in \{5,10\}$, $h = 1$ and $p \in \{5,10\}$. Here $g(N,M)$ denotes the minimum cost as a function of N (number of machines) and M (maximum batch size). The corresponding optimal policies are easily computed using the optimality equations (7.23), (7.14) and (7.20). The shorthand notation for the optimal policy π^* in Tables 7.1 and 7.2 was introduced in section 5.11; to repeat, n^m denotes a string of m n's, n-m denotes the string $n, n-1, \ldots, m$ $(m < n)$, $[n\text{-}m]^{i_n,\ldots,i_m}$ denotes $(n^{i_n}, \ldots, m^{i_m})$ $(m < n)$, and the remaining elements of π^* are zero. For the case $D = 1, L = 0$ (Table 7.3) this notation has to be extended, because there the single state variable denotes net inventory which may be negative (see Remark (i) in subsection 7.2.3). We let $(a; b, \pi_0)$ denote the policy π with $\pi(i) = b$ for $i \leq a$ and $\pi(i)$ for $i > a$ is given by π_0 (using the same notation as in Tables 7.1 and 7.2), e.g., $(-5; 10, 9^2, 8\text{-}6) := (\ldots, 10, 9, 9, 8, 7, 6, 0, \ldots)$ with $\pi(-5) = 10$. Finally, the double apostrophe " indicates that the optimal policy is identical to the policy one column to the left.

Apparently, binding capacity constraints may cause non-monotonicities in the optimal policy; see e.g. the instance $N = 1$, $M = 15$, $K = 10$, $p = 10$ in Table 7.1 as well as in Table 7.2. Also, if $N > 1$, $M < \infty$ and $K > 0$, the optimal policy often has jumps due to the set-up cost of K that is incurred for every batch of size at most M. As a matter of fact, the numerical results suggest that in these cases a so-called $(s_1, \ldots, s_n; Q)$-policy, characterized by

$$\pi(i) = \begin{cases} nQ & \text{if } i \leq s_n; \\ (n-1)Q & \text{if } s_n < i \leq s_{n-1}; \\ \cdots & \cdots \\ Q & \text{if } s_2 < i \leq s_1; \\ 0 & \text{if } i > s_1, \end{cases} \tag{7.25}$$

is likely to perform well. Clearly $Q = M$ and $n \leq N$, while s_i $(i = 1, \ldots, n)$ may be negative for the case $D = 1, L = 0$.

If a " sign appears in Tables 7.1 and 7.2 then at least one of the N machines is superfluous, i.e., it remains unused under the optimal policy. Obviously, the smaller the capacity M of a single machine and the smaller the set-up cost K, the higher the marginal value of an additional machine. On the other hand, if M is large enough then one or two machines may suffice in order to carry out the optimal policy. The sensitivity of the minimum cost to the capacity parameters N and M is nicely illustrated in Figures 7.1 and 7.2, where $g(N,M)$ is plotted against M for different values of N. For the case $D = 0$ and $L = 1$ with $K = 10$ and $p = 5$ (Figure 7.1), we see that a second machine is only used for $M = 3, \ldots, 8$ while a third machine is only used for $M = 3, 4$. For the case $D = 1$ and $L = 0$ (Figure 7.2) the transitions from $N = 3$ to $N = 2$ and $N = 2$ to $N = 1$ are "smoother", in the sense

K	p	M	$g(1,M)$	π^*	$g(2,M)$	π^*	$g(3,M)$	π^*
5	5	5	30.0476	(5^{25})	15.4126	$(10^6, 5^5)$	15.0716	$(15, 10^5, 5^5)$
		10	10.9452	$(10^5, 9\text{-}5)$	10.9452	"	10.9452	"
		15	9.9028	$(13\text{-}5)$	9.9028	"	9.9028	"
5	10	5	55.0476	(5^{50})	18.3406	$(10^9, 5^5)$	16.3742	$(15^2, 10^5, 5^5)$
		10	13.7179	$(10^8, 9\text{-}5)$	12.2214	$(15, 14, 10^4, 9\text{-}4)$	12.2214	"
		15	11.0402	$(14\text{-}4)$	11.0402	"	11.0402	"
10	5	5	35.0476	(5^{20})	24.6591	$(10^5, 5^5)$	24.6591	"
		10	15.7948	(10^9)	15.7948	"	15.7948	"
		15	14.4376	$(14\text{-}12, 15^4, 14)$	14.4376	"	14.4376	"
10	10	5	60.0476	(5^{45})	27.8272	$(10^9, 5^5)$	26.1496	$(15^2, 10^5, 5^5)$
		10	18.6394	$(10^9, 9, 10^2)$	17.7611	$(20, 10^7, 9, 8, 10)$	17.7611	"
		15	15.8268	$(15\text{-}10, 15^2, 14, 13)$	15.8268	"	15.8268	"

Table 7.1: Optimal policy under capacity constraints ($D = L = 0$)

K	p	M	$g(1,M)$	π^*	$g(2,M)$	π^*	$g(3,M)$	π^*
5	5	5	30.0476	(5^{35})	15.9527	$(10^{16}, 5^5)$	15.9527	"
		10	11.4815	$(10^{15}, 9\text{-}5)$	11.4815	"	11.4815	"
		15	11.1808	$(12^{11}, 11^2, 10^2, 9\text{-}5)$	11.1808	"	11.1808	"
5	10	5	55.0476	(5^{60})	18.8041	$(10^{20}, 5^5)$	18.0457	$(15^{13}, 10^6, 5^5)$
		10	14.1711	$(10^{18}, 9\text{-}5)$	13.7431	$(15^{10}, 14^2, 10^6, 9\text{-}5)$	13.7431	"
		15	12.8117	$([14\text{-}10]^{9,3,2,1,2}, 9\text{-}5)$	12.8117	"	12.8117	"
10	5	5	35.0476	(5^{30})	25.2021	$(10^{15}, 5^5)$	25.2021	"
		10	16.2690	(10^{19})	16.2690	"	16.2690	"
		15	15.5838	$(13^{10}, 12^3, 15^4, 14)$	15.5838	"	15.5838	"
10	10	5	60.0476	(5^{55})	28.3017	$(10^{19}, 5^5)$	27.7706	$(15^{12}, 10^6, 5^5)$
		10	19.0711	(10^{22})	19.0507	$(16^9, 15, 10^{12})$	19.0507	"
		15	17.3737	$([14\text{-}11]^{10,3,2,1}, [15\text{-}13]^{3,1,1})$	17.3737	"	17.3737	"

Table 7.2: Optimal policy under capacity constraints ($D = 0,\ L = 1$)

K	p	M	$g(1,M)$	π^*	$g(2,M)$	π^*	$g(3,M)$	π^*
5	5	5	30.0000	$(24; 5)$	11.8565	$(1; 10, 5^5)$	10.0805	$(-9; 15, 10^5, 5^5)$
		10	7.1617	$(0; 10, 9\text{-}6)$	5.1684	$(-11; 20, 10^{10})$	5.1664	$(-21; 30, 20^{10}, 10^{10})$
		15	4.2093	$(-5; 15, 14\text{-}11)$	4.1526	π_1	4.1526	π_2
5	10	5	55.0000	$(49;5)$	13.6396	$(3; 10, 5^5)$	10.1332	$(-8; 15, 10^5, 5^5)$
		10	8.8920	$(2; 10, 9\text{-}5)$	5.1708	$(-11; 20, 10^{10})$	5.1664	$(-21; 30, 20^{10}, 10^{10})$
		15	4.2916	$(-5; 15, 14\text{-}11)$	4.1526	π_1	4.1526	π_2
10	5	5	35.0000	$(19; 5)$	21.4622	$(0; 10, 5^5)$	20.0686	$(-9; 15, 10^5, 5^5)$
		10	12.0118	$(3; 10)$	10.1681	$(-11; 20, 10^{10})$	10.1664	$(-21; 30, 20^{10}, 10^{10})$
		15	7.6763	$(-3; 15, 14, 13)$	7.6343	π_3	7.6343	π_4
10	10	5	60.0000	$(44;5)$	23.3003	$(3; 10, 5^5)$	20.1271	$(-8; 15, 10^5, 5^5)$
		10	13.8125	$(6; 10)$	10.1705	$(-11; 20, 10^{10})$	10.1664	$(-21; 30, 20^{10}, 10^{10})$
		15	7.7533	$(-3; 15, 14, 13)$	7.6343	π_3	7.6343	π_4

$\pi_1 := (-20; 30, 29\text{-}26, 15^{11}, 14\text{-}11)$, $\pi_2 := (-35; 45, 44\text{-}41, 30^{11}, 29\text{-}26, 15^{11}, 14\text{-}11)$,
$\pi_3 := (-18; 30, 29, 28, 15^{13}, 14, 13)$, $\pi_4 := (-33; 45, 44, 43, 30^{13}, 29, 28, 15^{13}, 14, 13)$

Table 7.3: Optimal policy under capacity constraints ($D = 1,\ L = 0$)

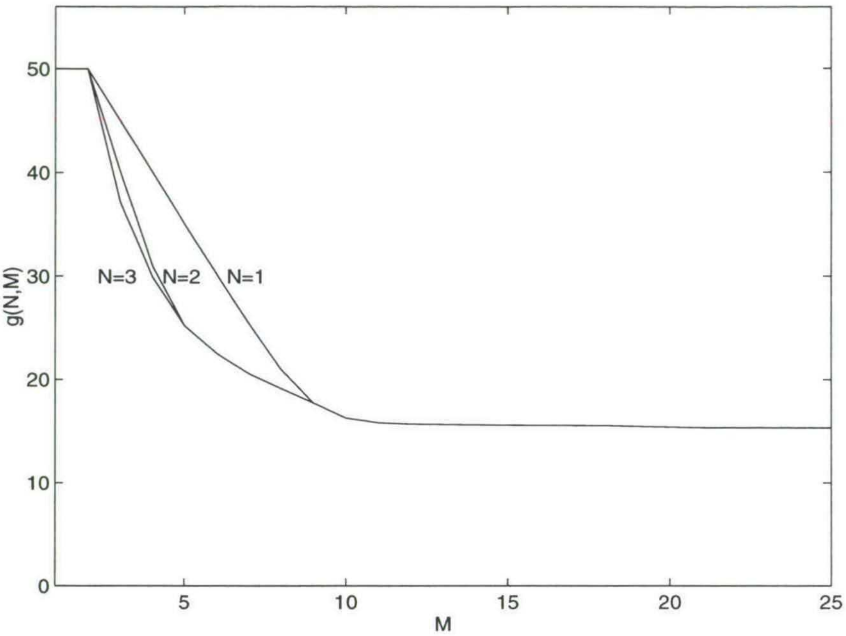

Figure 7.1: Minimum cost as a function of M for $N=1,2,3$ ($D=0$, $L=1$, $K=10$, $p=5$)

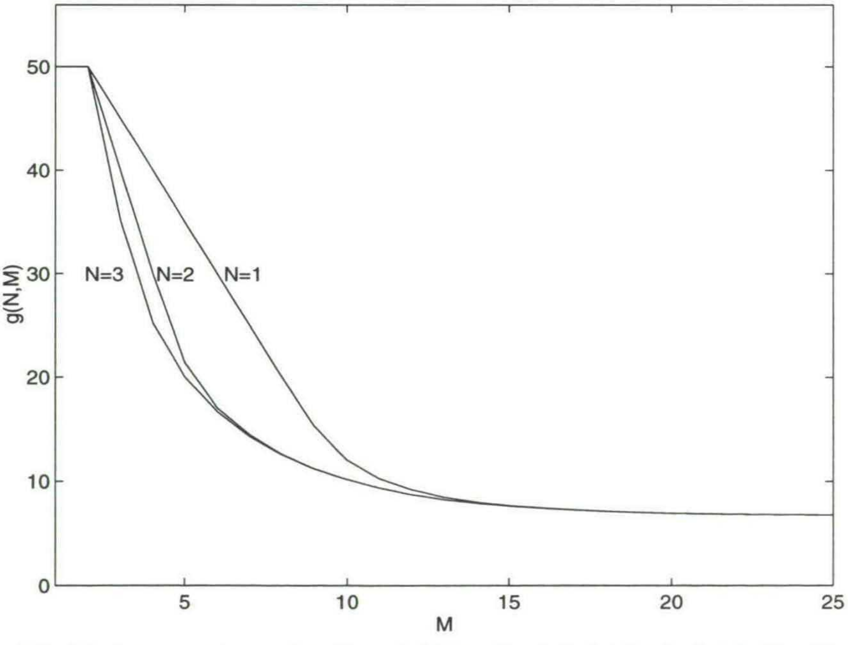

Figure 7.2: Minimum cost as a function of M for $N=1,2,3$ ($D=1$, $L=0$, $K=10$, $p=5$)

that $g(3, M)$ and $g(2, M)$ converge rapidly to, but never entirely coincide with, $g(1, M)$ for increasing M (see also the last column of Table 7.3, where the optimal policies require all three machines). Furthermore, $g(N, M)$ is a concave function of M for fixed N (whence $\min_N g(N, M)$ is also concave in M), except for very small values of M. Here a Never-Batch-Production policy (NBP; do not produce at all) is optimal with corresponding cost $p\mu$.

7.3 The (T, Q)-policy

It is obvious from the previous section that it is computationally infeasible to find an optimal policy for a multi-machine model ($N > 1$) even for moderate values of $D+L$, due to the $(\max\{D,1\}+\max\{L,1\}-1)$-dimensional state space. Indeed, even if it would be feasible then the resulting policy would be too complex to be of any practical use. It is therefore important to look for heuristic policies that perform well and are easy to compute. Heuristic policies can roughly be divided into two categories, namely dynamic policies and static policies. Under a dynamic policy the decisions depend on the state of the system (such as on-hand inventory or waiting demand), whereas under a static policy the decisions do not depend on the state of the system (a static policy can be seen as a dynamic policy where the same decision is taken in all states). Clearly, dynamic policies perform better than static policies, simply because they allow for more flexible control on the system. Unfortunately, however, almost all dynamic policies for the capacitated production/inventory model suffer from the same curse of dimensionality as the optimal policy. To compute the expected average costs for a given dynamic policy, like (s, Q) or (s, S), usually requires exactly the same multi-dimensional state description as the MDP. Exceptions to this rule are the Critical-Group policy for the service model (see section 2.3) and the Critical-Group-Production policy for the production/inventory model with $D > L$ and $N = \infty$ (see subsection 6.3.1), for which it suffices to keep track of the "critical group" only (demand with a residual delay-limit of zero). So, as almost all of the dynamic heuristic policies are computationally infeasible whenever the optimal policy is, we go one step further back and resort to static heuristic policies. Besides theoretical reasons, static policies are also important for a number of practical reasons. First of all, static policies are easy to understand and implement, as they are completely predictable and plannable. Under a static policy the epochs at which a production batch is started are known in advance, while under a dynamic policy they depend on the state of system and are stochastic. Secondly, complete state information may not be available or require costly inspections. To put it differently, dynamic policies are only implementable if there is an information system that reveals the state of system either continuously or at fixed review points.

We focus on the most sensible static policy: start a production batch of Q items every T periods. We refer to this policy as the (T, Q)-policy. The (T, Q)-policy is very flexible, in the sense that it can be employed for any value of D and L. The capacity parameters N and M are taken into account a priori, by restricting the choice of the policy parameters T and Q. To determine the number of outstanding batches for a given (T, Q)-policy, suppose that a new batch is started at the beginning of period 1. When this batch is completed at the end of period L, exactly $\lfloor\frac{L}{T}\rfloor$ new batches have been started in the mean time. It follows that $\lfloor\frac{L}{T}\rfloor+1$ batches are outstanding in periods $\lfloor\frac{L}{T}\rfloor T+1,\ldots,L$ and $\lfloor\frac{L}{T}\rfloor$ batches in $L+1,\ldots,(\lfloor\frac{L}{T}\rfloor+1)T$. For example, if $L = 7$ and $T = 2$ then 4 batches are outstanding in period 7 and 3 batches in period 8, and if $L = 6$ and $T = 2$ then 3 batches are outstanding in periods 7 and 8. The same argument holds for any subsequent review interval $(n-1)T+1,\ldots,nT$, so that $\lfloor\frac{L}{T}\rfloor+1$ batches are outstanding during a fraction of $\frac{L}{T}-\lfloor\frac{L}{T}\rfloor$ of the time and $\lfloor\frac{L}{T}\rfloor$ batches during a fraction of $\lfloor\frac{L}{T}\rfloor+1-\frac{L}{T}$ of the time. Consequently, the policy parameters are bounded by

$$\left\lceil\frac{L}{T}\right\rceil \leq N,\ Q \leq M \quad\Longleftrightarrow\quad T \geq \left\lceil\frac{L}{N}\right\rceil,\ Q \leq M. \tag{7.26}$$

Remark. The capacity constraints (7.26) assume that exactly one machine is started every T periods, but we can easily incorporate the possibility to start multiple machines simultaneously every T periods. An extreme example of such a policy is to start a production run for NM items on N machines every L periods. More generally, under a (T, Q)-policy with $Q \leq NM$, a production run for Q items on $\lceil\frac{Q}{M}\rceil$ machines is started every T periods. If we group the $\lceil\frac{Q}{M}\rceil$ machines with set-up cost K into one "super-machine" with set-up cost $K\lceil\frac{Q}{M}\rceil$, then exactly one "super-machine" is started every T periods. Since there are $\left\lfloor\frac{N}{\lceil\frac{Q}{M}\rceil}\right\rfloor$ "super-machines", the capacity constraints generalize to

$$\left\lceil\frac{L}{T}\right\rceil \leq \left\lfloor\frac{N}{\lceil\frac{Q}{M}\rceil}\right\rfloor,\ Q \leq NM \quad\Longleftrightarrow\quad T \geq \left\lceil\frac{L}{\left\lfloor\frac{N}{\lceil\frac{Q}{M}\rceil}\right\rfloor}\right\rceil,\ Q \leq NM. \tag{7.27}$$

Consequently, we may assume w.l.o.g. in the following that exactly one (super-)machine is started every T periods.

We now compute the expected average costs for a given (T, Q)-policy by using an embedded Markov chain on the epochs that a production batch has just been completed. Obviously, the transition time between two embedding epochs is constant and equal to T periods (whence the process reduces to a discrete-time Markov chain). The resulting Markov chain is only ergodic (i.e., a stationary distribution exists) if

$$Q < T\mu \quad\Longleftrightarrow\quad \frac{Q}{T} < \mu, \tag{7.28}$$

since on-hand inventory tends to infinity if the average demand per review period ($T\mu$) is smaller than or equal to Q. We must distinguish between the cases $D \leq T$ and $D > T$.

The case $D \leq T$

If $D \leq T$ then all demand not satisfied at an embedding epoch is immediately lost, so that the state description is given by

$I_n :=$ on-hand inventory just after the n^{th} production batch is completed $(n = 1, 2, \ldots)$.

Since the demand in periods $(n-1)T+1,\ldots,nT-D$ *cannot* be backordered from the incoming batch at time nT, while the demand in periods $nT-D+1,\ldots,nT$ *can*, we have that

$$I_n = \Big((I_{n-1} - S_{(n-1)T+1,nT-D})^+ + Q - S_{nT-D+1,nT}\Big)^+ \quad (n = 1,2,\ldots), \tag{7.29}$$

where $I_0 := 0$. This is exactly the same transition equation that appeared in the SMDP of Chapter 5 for $a > 0$, provided that we set $L = T$, $i = I_{n-1}$, $a = Q$ and $n = 1$ (see (5.27)). In other words, $\{I_n\}$ is the embedded Markov chain induced by the SMDP of Chapter 5 upon setting $R_i = Q$ for all i (and $L = T$). Therefore the transition probabilities of $\{I_n\}$ are simply given by

$$p_{ij} := \Pr\{I_n = j \mid I_{n-1} = i\} = p_{ij}(Q \mid L = T) \quad (i,j = 0,1,\ldots), \tag{7.30}$$

where $p_{ij}(Q \mid L = T)$ denotes the transition probability $p_{ij}(Q)$ in (5.28) with L replaced by T. Suppose that the stationary distribution of $\{I_n\}$ is $\{\pi_i(T,Q),\ i \geq 0\}$, then the expected average costs are given by

$$g_1(T,Q) = \sum_{i=0}^{\infty} \pi_i(T,Q) c_i(Q \mid L = T) \quad (T \geq D), \tag{7.31}$$

where $c_i(Q \mid L = T)$ denotes the transition costs $c_i(Q)$ in (5.29) with L replaced by T.

The case $D > T$

If $D > T$ then waiting demand with a residual delay-limit of T or more periods can still be satisfied at the next embedding epoch. More generally, demand with a residual delay-limit between jT and $(j+1)T-1$ periods can still be satisfied until the j^{th} next embedding epoch ($j = 1,\ldots,\lceil\frac{D}{T}\rceil-1$). This requires additional state variables

$R_{n,j}$:= number of demands that can be satisfied until the j^{th} next embedding epoch just after the n^{th} production batch is completed $(n \geq 1;\ j = 1,\ldots,\lceil\frac{D}{T}\rceil - 1)$,

so that the state space becomes

$$\Omega := \Big\{i \mid i = 0,1,\ldots\Big\} \bigcup \Big\{(r_1,\ldots,r_{\lceil\frac{D}{T}\rceil-1}) \mid r_j \in I\!N,\ j = 1,\ldots,\lceil\tfrac{D}{T}\rceil - 1\Big\}. \tag{7.32}$$

Since either $I_n = 0$ or $R_{n,1} = \cdots = R_{n,\lceil\frac{D}{T}\rceil-1} = 0$ the dimension of Ω is $\lceil\frac{D}{T}\rceil - 1$. As a result, the expected average costs of the (T,Q)-policy can only be computed effectively for small values of $\lceil\frac{D}{T}\rceil$. At first sight it seems that even the static (T,Q)-policy suffers from the curse of dimensionality, but it turns out the case $\lceil\frac{D}{T}\rceil > 2$ can often be excluded from consideration without substantial loss of generality. Obviously,

$$\left\lceil\frac{D}{T}\right\rceil = m \iff \frac{D}{m} \leq T < \frac{D}{m-1} \quad (m = 2,3,\ldots). \tag{7.33}$$

For example, if $D = 5$ then $\lceil\frac{D}{T}\rceil = 2$ for $T \in \{3,4\}$, $\lceil\frac{D}{T}\rceil = 3$ for $T = 2$ and $\lceil\frac{D}{T}\rceil = 5$ for $T = 1$. Hence the case $\lceil\frac{D}{T}\rceil > 2$ corresponds to T-values in the region $\{T : 0 < T < \frac{D}{2}\}$, and it turns out that the optimal value of T seldom lies in this region (notice that for $D \leq 2$ this region is empty). Furthermore, we have by the capacity constraint $T \geq \frac{L}{N}$ (see (7.26)) that

$$\left\lceil \frac{D}{T} \right\rceil \leq \left\lceil \frac{DN}{L} \right\rceil, \tag{7.34}$$

so that we can assume w.l.o.g. that $\lceil \frac{D}{T} \rceil \leq 2$ if $\frac{DN}{L} \leq 2$. For these reasons we only consider the case $\lceil \frac{D}{T} \rceil = 2$ here (or $\frac{D}{2} \leq T < D$).

Remark. To compute the expected average costs for $\lceil \frac{D}{T} \rceil > 2$, it is actually not necessary to use the $(\lceil \frac{D}{T} \rceil - 1)$-dimensional state space (7.32). An alternative approach is to use an embedded Markov chain on epochs that on-hand inventory is positive. This reduces the dimension of the state space to 1, but now the difficulty lies in the computation of the transition probabilities, the transition times and the transition costs. Although this approach is computationally more attractive, the analysis is rather cumbersome and we will not go into detail here.

For $\lceil \frac{D}{T} \rceil = 2$ the state vector is $(I_n, R_{n,1})$, leading to a one-dimensional state space as either $I_n = 0$ or $R_{n,1} = 0$. Therefore we use the state variable $\tilde{I}_n := I_n - R_{n,1}$, which can be interpreted as "net inventory" (inventory on hand minus backorders; see also Remark (i) in subsection 7.2.3). If $\tilde{I}_n \geq 0$ then on-hand inventory is $\tilde{I}_n$ and there is no waiting demand, while if $\tilde{I}_n < 0$ then on-hand inventory is zero and there are $-\tilde{I}_n$ waiting demands (that can only be backordered from the next production batch). Since at most Q demands can be backordered, the state space becomes

$$\Omega := \{ i \mid i = -Q, -Q+1, \ldots, -1, 0, 1, \ldots \}. \tag{7.35}$$

Now define

$$X^{(n)} := S_{(n-1)T+1,nT}, \quad X_l^{(n)} := S_{(n-1)T+1,(n+1)T-D}, \quad X_r^{(n)} := S_{(n+1)T-D+1,nT}, \tag{7.36}$$

so that $X^{(n)}$ denotes the total demand in the n^{th} review period of which $X_l^{(n)}$ (the demand in the first $2T-D$ periods) *cannot* and $X_r^{(n)}$ (the demand in the last $D-T$ periods) *can* be backordered at the $(n+1)^{\text{th}}$ embedding epoch. It follows that

$$\tilde{I}_n = \begin{cases} \max\{\tilde{I}_{n-1} + Q - X^{(n)}, -Q\} & \text{if } X_l^{(n)} \leq \tilde{I}_{n-1} + Q \\ \max\{-X_r^{(n)}, -Q\} & \text{if } X_l^{(n)} > \tilde{I}_{n-1} + Q \end{cases} \quad (n = 1, 2, \ldots), \tag{7.37}$$

or, equivalently,

$$\tilde{I}_n = \max\{(\tilde{I}_{n-1} + Q - X_l^{(n)})^+ - X_r^{(n)}, -Q\} \quad (n = 1, 2, \ldots). \tag{7.38}$$

Hence the transition probabilities of $\{\tilde{I}_n\}$ are given by

$$p_{ij} = \begin{cases} \displaystyle\sum_{k=0}^{i+Q-1} q_k^{(2T-D)} \bar{Q}_{i+2Q-k}^{(D-T)} + \bar{Q}_{i+Q}^{(2T-D)} \bar{Q}_Q^{(D-T)} & \text{if } j = -Q \\ \displaystyle\sum_{k=0}^{i+Q-1} q_k^{(2T-D)} q_{i+Q-k-j}^{(D-T)} + \bar{Q}_{i+Q}^{(2T-D)} q_{-j}^{(D-T)} & \text{if } -Q < j \leq 0 \\ \displaystyle\sum_{k=0}^{i+Q-j} q_k^{(2T-D)} q_{i+Q-k-j}^{(D-T)} & \text{if } 0 < j \leq i + Q \end{cases} \quad (i \in \Omega). \tag{7.39}$$

Next we turn to the one-step costs. Let $c_h(i)$ and $c_p(i)$ denote the holding and the penalty costs, respectively, until the next embedding epoch when net inventory is i. Using (5.30) we find that

$$c_h(i) = \begin{cases} 0 & \text{if } -Q \le i \le 0; \\ E\{\sum_{n=1}^{T} h(i-S_n)^+\} = h\sum_{n=1}^{T}\sum_{k=0}^{i-1} Q_k^{(n)} & \text{if } i > 0. \end{cases} \tag{7.40}$$

Penalty costs are incurred when $X_l^{(n)} > i+Q$ and when $X_r^{(n)} > (i+Q-X_l^{(n)})^+ + Q$ (i.e., when backorders exceed Q), and using (5.31) it follows that

$$c_p(i) = E\{p(X_l^{(n)} - (i+Q))^+\} + \sum_{k=0}^{i+Q-1} q_k^{(2T-D)} E\{p(X_r^{(n)} - (i+2Q-k))^+\} +$$
$$\bar{Q}_{i+Q}^{(2T-D)} E\{p(X_r^{(n)} - Q)^+\} = p\Big(T\mu - \sum_{k=1}^{i+Q} \bar{Q}_k^{(2T-D)} -$$
$$\sum_{k=0}^{i+Q-1} q_k^{(2T-D)} \sum_{l=1}^{i+2Q-k} \bar{Q}_l^{(D-T)} - \bar{Q}_{i+Q}^{(2T-D)} \sum_{k=1}^{Q} \bar{Q}_k^{(D-T)}\Big) \quad (i \in \Omega) \tag{7.41}$$

Remark. For $D = 2T$ the above formulas remain valid by setting $q_0^{(0)} := 1$. In this case (7.39) and (7.41) simplify to

$$p_{ij}(a) = \begin{cases} \bar{Q}_{i+2Q}^{(T)} & \text{if } j = -Q \\ q_{i+Q-j}^{(T)} & \text{if } -Q < j \le i+Q \end{cases} \quad (i \in \Omega) \tag{7.42}$$

and

$$c_p(i) = p\Big(T\mu - \sum_{k=1}^{i+2Q} \bar{Q}_k^{(T)}\Big) \quad (i \in \Omega), \tag{7.43}$$

respectively.

Suppose that the stationary distribution of $\{\tilde{I}_n\}$ is $\{\pi_i(T,Q),\ i \in \Omega\}$, then the expected average costs per period are given by

$$g_2(T,Q) = \frac{1}{T}\Big(K + \sum_{i\in\Omega} \pi_i(T,Q)\big(c_h(i) + c_p(i)\big)\Big) \quad (\tfrac{D}{2} \le T < D), \tag{7.44}$$

with $c_h(i)$ and $c_p(i)$ given by (7.40) and (7.41), respectively.

7.3.1 Numerical results

Let $g_m(T,Q)$ denote the expected average costs for a (T,Q)-policy with $\lceil \frac{D}{T} \rceil = m$, then the optimal (T,Q) pair follows from

$$g(T^*,Q^*) := \min_{(T,Q):T\ge\lceil \frac{L}{N}\rceil, Q\le M} g_{\lceil \frac{D}{T}\rceil}(T,Q) \tag{7.45}$$

As argued earlier, we restrict the search for T^* to the region $\{T : T \ge \frac{D}{2}\}$, for which we only need $g_1(T,Q)$ (see (7.31)) and $g_2(T,Q)$ (see (7.44)). We use the following search algorithm:

1 $T := \max\{\lceil \frac{L}{N} \rceil, \lceil \frac{D}{2} \rceil\}$, $Q := \min\{\lceil T\mu \rceil - 1, M\}$, $g_{\text{new}} := \infty$, $Q_{\text{new}} := 0$;

2 $\quad g_{\text{old}} := g_{\text{new}}$, $Q_{\text{old}} := Q_{\text{new}}$, $g := \infty$;

3 $\qquad g' := g$, $g := g_{\lceil \frac{D}{T} \rceil}(T,Q)$;

4 $\qquad$ if $g < g'$, $Q := Q-1$, go to 3;

5 $\quad g_{\text{new}} := g'$, $Q_{\text{new}} := Q-1$;

6 $\quad$ if $g_{\text{new}} < g_{\text{old}}$, $T := T+1$, go to 2;

7 $T^* := T-1$, $Q^* := Q_{\text{old}}$, $g^* := g_{\text{old}}$.

This search algorithm implicitly assumes that the cost function $g_{\lceil \frac{D}{T} \rceil}(T,Q)$ is unimodular both in T and in Q. Although we were not able to prove this, this is strongly supported by numerical evidence.

Remark. The above search algorithm finds the optimal (T,Q)-policy for a given number of simultaneously started machines n (see also the first Remark in section 7.3). To find the optimal value of n, the algorithm must be applied repeatedly for $n = 1, \ldots, N$.

In Table 7.4 we compute T^* and Q^* using the above search procedure, for $D \in \{0, \ldots, 5\}$, $\lceil \frac{L}{N} \rceil = \lceil \frac{D}{2} \rceil$, $M \in \{10, 20, \infty\}$, X_n Poisson distributed with mean $\mu \in \{5, 10\}$, $K \in \{10, 50\}$, $h = 1$ and $p \in \{5, 10\}$. We make the following observations:

- Setting $\lceil \frac{L}{N} \rceil = \lceil \frac{D}{2} \rceil$ guarantees that $\lceil \frac{D}{T} \rceil \leq 2$, so that the search procedure will lead to the optimal (T,Q)-policy subject to the capacity constraints $T \geq \lceil \frac{L}{N} \rceil$ and $Q \leq M$.

- If the optimal (T,Q)-policy for $M = \infty$ is feasible for $M = 20$ c.q. $M = 10$, then it is also optimal for $M = 20$ c.q. $M = 10$.

- For $M = \infty$ the optimal value of T always lies in the region $\{T : T \geq D\}$, except for the instance $D = 5$, $\mu = 10$, $K = 10$, $p = 5$. Consequently, we can safely confine ourselves to values of $\lceil \frac{D}{T} \rceil \leq 2$ for large values of M.

- If the production frequency T tends to infinity, the (T,Q)-policy reduces to a NBP-policy, i.e.,
$$\lim_{T \to \infty} g_{\lceil \frac{D}{T} \rceil}(T,Q) = \lim_{T \to \infty} g_1(T,Q) = g_{\text{NBP}} = p\mu. \tag{7.46}$$
Hence if $(T^*, Q^*) = (\infty, 0)$ in Table 7.4 then a NBP-policy is optimal. This indicates that the production capacity is insufficient and the set-up cost too high to justify production.

- If $(T^*, Q^*) = (\lceil \frac{L}{N} \rceil, M)$ then the production capacity is fully utilized (the capacity constraints are "binding"). In particular, this will be the case whenever the average demand per period (μ) is considerably larger than the average production capacity per period ($\frac{NM}{L}$). Then the expected holding costs are close to zero, while on average

D	$\lceil \frac{L}{N} \rceil$	μ	K	p	$M = 10$ $g(T^*, Q^*)$		$M = 20$ $g(T^*, Q^*)$		$M = \infty$ $g(T^*, Q^*)$	
0	1	5	10	5	12.5255	(2,8)	12.2230	(3,13)	12.2230	(3,13)
				10	15.2481	(2,9)	15.2481	(2,9)	15.2481	(2,9)
			50	5	25	(∞,0)	21.9085	(5,20)	21.7257	(6,23)
				10	35.2481	(2,9)	26.3016	(4,18)	25.5640	(5,22)
		10	10	5	18.2355	(1,9)	16.7231	(2,18)	16.7231	(2,18)
				10	23.2355	(1,9)	21.7231	(2,18)	21.7231	(2,18)
			50	5	50	(∞,0)	36.7231	(2,18)	31.4194	(4,35)
				10	63.2355	(1,9)	41.7231	(2,18)	36.3382	(4,37)
1	1	5	10	5	9.7450	(2,9)	9.1292	(3,13)	9.1292	(3,13)
				10	12.2450	(2,9)	12.2450	(2,9)	12.2450	(2,9)
			50	5	25	(∞,0)	18.8196	(5,20)	18.5246	(6,25)
				10	32.2450	(2,9)	22.6406	(4,18)	21.9250	(5,22)
		10	10	5	15.4779	(1,9)	11.5650	(2,18)	11.5650	(2,18)
				10	20.4779	(1,9)	15.1479	(2,19)	15.1479	(2,19)
			50	5	50	(∞,0)	31.5650	(2,18)	24.7310	(4,36)
				10	60.4779	(1,9)	35.1479	(2,19)	28.9179	(4,37)
2	1	5	10	5	8.3662	(2,9)	7.5018	(3,13)	7.5018	(3,13)
				10	10.8662	(2,9)	9.8646	(3,14)	9.8646	(3,14)
			50	5	25	(∞,0)	16.7129	(5,20)	15.8812	(6,26)
				10	30.8662	(2,9)	20.0616	(4,18)	18.7848	(6,27)
		10	10	5	15.0693	(1,9)	9.4393	(2,19)	8.7598	(3,28)
				10	20.0693	(1,9)	11.9393	(2,19)	11.9393	(2,19)
			50	5	50	(∞,0)	29.4393	(2,19)	20.0822	(4,37)
				10	60.0693	(1,9)	31.9393	(2,19)	23.7161	(4,38)
3	2	5	10	5	7.8300	(2,9)	6.0904	(4,18)	6.0904	(4,18)
				10	10.3300	(2,9)	8.3450	(3,14)	8.3450	(3,14)
			50	5	25	(∞,0)	15.5663	(5,20)	13.8307	(6,26)
				10	30.3300	(2,9)	18.5904	(4,18)	16.4134	(6,27)
		10	10	5	30.0000	(2,10)	8.2256	(2,19)	7.2467	(3,28)
				10	55.0000	(2,10)	10.7256	(2,19)	10.1199	(4,38)
			50	5	50	(∞,0)	28.2256	(2,19)	16.6182	(5,46)
				10	75.0000	(2,10)	30.7256	(2,19)	19.7946	(5,47)
4	2	5	10	5	7.6256	(2,9)	5.4254	(4,18)	5.3454	(5,23)
				10	10.1256	(2,9)	7.5417	(3,14)	7.3454	(5,23)
			50	5	25	(∞,0)	15.1346	(5,20)	12.0916	(7,31)
				10	30.1256	(2,9)	17.5543	(4,19)	14.4564	(7,32)
		10	10	5	30.0000	(2,10)	7.7715	(2,19)	5.9926	(4,38)
				10	55.0000	(2,10)	10.2715	(2,19)	8.4926	(4,38)
			50	5	50	(∞,0)	27.7715	(2,19)	14.2637	(6,56)
				10	75.0000	(2,10)	30.2715	(2,19)	17.1503	(5,48)
5	3	5	10	5	11.6667	(3,10)	5.1623	(4,18)	4.6284	(5,23)
				10	20.0000	(3,10)	6.5628	(4,19)	6.6018	(6,28)
			50	5	25	(∞,0)	15.0249	(5,20)	10.8042	(8,36)
				10	33.3333	(3,10)	16.5628	(4,19)	13.0193	(8,37)
		10	10	5	36.6667	(3,10)	20.0000	(3,20)	5.3714	(4,38)
				10	70.0000	(3,10)	36.6667	(3,20)	7.4433	(5,48)
			50	5	50	(∞,0)	33.3333	(3,20)	12.4587	(6,57)
				10	83.3333	(3,10)	50.0000	(3,20)	14.9587	(6,57)

Table 7.4: Numerical evaluation of the (T, Q)-policy

Q demands are satisfied from production and $\mu T - Q$ demands are lost. Hence the expected average costs can be approximated by

$$g(T,Q) \approx \frac{K + p(\mu T - Q)}{T} = p\mu - \frac{pQ - K}{T} \quad (\mu > \tfrac{NM}{L}), \tag{7.47}$$

which explains the round values for $g(T,Q)$ in Table 7.4.

In order to address the performance of the (T,Q)-policy with respect to other (dynamic) policies, we can compare Table 7.4 to Table 5.2 (where $D \leq L$, $N=1$, $M=\infty$) as well as to Tables 6.1 and 6.2 (where $D > L$, $N = M = \infty$). It turns out that for $D \leq L$ the (T,Q)-policy performs remarkably well and is close to the optimal policy in some cases, e.g., for $D=0$, $L=1$, $\mu=5$, $K=50$, $p=5$ the difference in cost is only 3.4% (see Table 5.2). Not surprisingly, the (T,Q)-policy does well when both the production frequency and the production quantities under an optimal policy have little variability. For $D > L$ the performance of the (T,Q)-policy is significantly worse, and here a CGP-policy (see section 6.3.1) is usually a better option.

7.4 An extended model including switch-over times

The general model of section 7.3 assumes that a machine is immediately available after having completed a production batch. However, one can also think of situations where a machine that has just completed a batch is only available for the next batch after a given switch-over time, needed to carry out maintenance, clean the machine or prepare the machine for the next batch. In this section we take a brief look at an extension of the model that includes switch-over times.

A particularly interesting problem also covered by this extension is the following "vehicle planning" problem. Consider a supplier of consumer items who transports items from a central location (e.g., a warehouse or depot) to a location where demand for the item arrives (e.g., a store or wholesaler). The supplier has N trucks each having a capacity of M items, and the travel time between the two locations is L periods. This problem does not fit the general model because the truck has to drive back to the depot, and hence is not available for $2L$ time units. In other words, the "transportation lead time" is $2L$ time units and items are not delivered at the end of the lead time but halfway through the lead time. However, it is easily seen that this model is equivalent to a production/inventory model with a production lead time of L time units (corresponding to the travel time from depot to wholesaler) and a switch-over time of L time units (corresponding to the travel time from the wholesaler back to the depot). The set-up cost K represents the fixed costs associated with every trip (e.g., driver and fuel costs).

We now show how the general periodic-review model of section 7.3 can be extended to the case where the switch-over times are constant and equal to S periods. To do so, we have to modify the state description and also keep track of the number of machines with a residual switch-over time of n periods ($n = 1, \ldots, S$). For ease of exposition we only illustrate this for the case $D = 0$, noting that the same modifications apply to the case $D > 0$.

A complete state description for this model is given by

$$\mathbf{z} := (i; j_1, \ldots, j_{L-1}; l_1, \ldots, l_S), \tag{7.48}$$

with

i := on-hand inventory at the start of a period;
j_n := number of items that will be completed in n periods $(n = 1, \ldots, L-1)$;
l_n := number of machines with a residual switch-over time of n periods $(n = 1, \ldots, S)$.

Since the number of machines needed to produce j items is $\lceil \frac{j}{M} \rceil$, the number of unavailable machines in state $\mathbf{z}$ is

$$N(\mathbf{z}) = \sum_{n=1}^{L-1} \left\lceil \frac{j_n}{M} \right\rceil + \sum_{n=1}^{S} l_n. \tag{7.49}$$

Consequently, the state space and action spaces are given by (7.13) and (7.4) respectively, with $\mathbf{z}$ and $N(\mathbf{z})$ given by (7.48) and (7.49) respectively. As for the transfer function, suppose that at a decision epoch the system is in state $\mathbf{z}$. Then at the next decision epoch j_1 items are completed, and $\lceil \frac{j_1}{M} \rceil$ machines start their switch-over time and are available S periods later. Hence the transfer function is given by

$$T(\mathbf{z}; j_L; k) = \left((i-k)^+ + j_1; j_2, \ldots, j_L; l_2, \ldots, l_S, \lceil \tfrac{j_1}{M} \rceil\right). \tag{7.50}$$

Finally, using (7.48), (7.49) and (7.50), the optimality equations are identical to (7.14).

Chapter 8

Conclusions and further research

In Part II of this thesis we have laid down the foundations for a broad class of production/inventory models, in the form of the general framework of Chapter 4. The building blocks for this framework are provided by the service model of Part I, in which a key role is played by the prespecified delay-limit D. Whereas the service model applies to situations where customer demand cannot be prepared in advance, the general framework applies to a production environment where a single (exchangeable) item can be produced in advance of demand, thereby generating serviceable inventory. Demand not satisfied within the delay-limit is lost or expedited against a fixed cost per item, corresponding to individual service against a fixed cost per customer in the service model. The two key model parameters are the constant delay-limit D and the constant production lead time L, and these give rise to the following dichotomy: if $D \leq L$ then demand can only be satisfied by producing in advance, while if $D > L$ then it is possible to postpone production until $D-L$ time units after demand arrival ("production to order"). The case $D \leq L$ (Chapter 5) leads to inventory-type models, where D can be seen as a time-limit on backorders; in particular, the case $D = 0$ corresponds to a lost-sales inventory model with order lead time L (see section 4.3.1 for a survey). The case $D > L$ (Chapter 6) is more related to the service model of Part I; a queue of waiting demand builds up until it is decided to start a production run that includes all waiting demand, and possibly a production surplus in anticipation of future demand. The policies for the service model are easily generalized to this case by replacing D with $D-L$ and adding a parameter for the excess production.

The general framework also incorporates capacity restrictions on the maximal number of simultaneous production batches or identical machines (N) and the maximal size of a production batch (M). In Chapter 7 we have studied a general "capacitated" periodic-review model that can be used for any combination of the model parameters D, L, N and M. The optimal production policy for this model is a complex function of on-hand inventory, the number of demands with a residual delay-limit of i periods ($i = 1, \ldots, D-1$) and the number of items that will be completed in j periods ($j = 1, \ldots, L-1$). Due to the $(L+D-1)$-dimensional state description, this model is only useful for moderate values of $L+D$. Fortunately, however, it is always possible to increase the period length (i.e., the review interval) such that $\max\{D, L\} \leq 3$; see also the discussion in section 2.8. A simple but versatile heuristic policy is the (T, Q)-policy: produce Q items every T time units. For this periodic policy the capacity constraints have no influence on the computation of the expected average costs, but only on the feasible region for T and Q. In general, the tighter the capacity constraints are, the better the performance of the (T, Q)-policy is. If the average production capacity per time unit ($\frac{NM}{L}$) is considerably smaller than the average demand per time unit (μ), then the optimal policy is either a (T, Q)-policy with $T = L$ and $Q = NM$ or a NBP-policy (do not produce at all).

As we have seen, different combinations of the model parameters D, L, N and M may lead to substantially different models. This makes it very difficult to give general rules for good production policies; different parameter settings call for conceptually different policies. The most obvious example is the dichotomy between $D \leq L$ and $D > L$, which is also reflected in the relevant policies. For $D \leq L$ well-known policies from inventory theory can be used, while for $D > L$ the policies for the service model can be extended. The (T, Q)-policy can be used for any parameter setting, but this may come at the expense of a poor performance. In Table 8.1 we give an overview of the various models and policies considered in Part II of this thesis. Since this is only a selection of models within the general framework, a lot of research remains to be done.

model	review	D, L	N, M	policy	section
P	periodic	$D = 0,\ L > 0$	$N = 1,\ M = \infty$	optimal	5.2+5.3.1
C	continuous	"	"	"	5.2+5.3.2
PB	periodic	$0 < D \leq L$	"	"	5.2+5.4.1
CB	continuous	"	"	"	5.2+5.4.2
P	periodic	$D = 0,\ L > 0$	"	(s, Q)	5.7
"	"	"	"	(s, S, Q)	5.8
PU	periodic	$D > L \geq 0$	$N = M = \infty$	optimal	6.2
"	"	"	"	CGP	6.3.1
"	"	"	"	TDP	6.3.2
"	"	"	"	ETDP	6.3.3
PC	"	"	$N = 1,\ M = \infty$	optimal	6.5
PU	continuous	"	$N = M = \infty$	TDP	6.6
general	periodic	$D > 0,\ L > 0$	$N \leq \infty,\ M \leq \infty$	optimal	7.2.1
"	"	$D = 0,\ L > 0$	"	"	7.2.2
"	"	$D > 0,\ L = 0$	"	"	7.2.3
"	"	$D = 0,\ L = 0$	"	"	7.2.4
"	"	$D \geq 0,\ L \geq 0$	"	(T, Q)	7.3

Table 8.1: Overview of the various models and policies considered in Part II

Besides other models within the framework, future research could focus on some relevant model extensions outside the framework. These include:

- We assume that the production lead time L is constant. It is also interesting to investigate production/inventory models with stochastic lead times, although this would complicate matters considerably – at least for multi-machine models – due to the phenomenon of "order crossing" (a production batch that is started later than another batch may be completed earlier).

- We assume that the production lead time is independent of the batch size, and, related to this, we assume that all items become serviceable at the end of the lead time (similar to ordering inventory). A different class of production/inventory models results if it is assumed that items become serviceable "one-for-one" at fixed intervals, so that the total lead time is proportional to the batch size.

- We assume that a set-up cost K is incurred for *every* production batch. However, one can also think of situations where a set-up cost is only incurred when the machine has been idle for a while, i.e., the set-up cost is zero when a new production run is started immediately after the previous run is completed. This would stimulate consecutive production batches and regular production patterns.

- We assume that the penalty costs are proportional to the number of "lost sales" (demands whose delay-limit has expired). However, if lost sales correspond to demands that are satisfied by other means ("emergency ordering"), there may well be a fixed cost involved with these lost sales. One of the few papers incorporating a fixed penalty cost is [Aneja&Noori 1987], who consider a periodic-review lost-sales (i.e., $D = 0$) inventory model with instantaneous deliveries (i.e., $L = 0$) where penalty costs of $B + pi$ are incurred when i demands are lost. Under certain conditions on the demand density, they prove that for this model an optimal (s, S)-policy exists. However, [Çetinkaya&Parlar 1996] point out that their model formulation is ambiguous and that there is a flaw in their proof, and they present an alternative model formulation and an alternative optimality proof. For a more general model with $D > 0$ and/or $L > 0$, or with $N < \infty$ and/or $M < \infty$, the consequences of a fixed penalty cost are not clear. We also note that there is an interesting equivalence with a two-supplier inventory model; one supplier with a lead time of L and ordering cost of $K + ci$ for i items and the other with a lead time of 0 and ordering cost of $B + pi$ for i items. For an example of a two-supplier model with full backordering of demand see [Janssen&de Kok 1998].

- We assume that no waiting costs are charged before expiration of the delay-limit. If waiting costs are incurred, e.g., waiting costs of w per waiting demand per unit of time, then it may occasionally be better to incur the penalty costs before the delay-limit has expired (provided that demand can be satisfied by other means). This gives rise to a different type of policy with an additional decision concerning the timing of lost sales within the delay-limit.

- We assume ample inventory capacity, i.e., on-hand inventory is not bounded from above. This is usually a reasonable assumption, since most policies naturally induce a maximal stock level (e.g., $s+Q$ under an (s, Q)-policy and S under an (s, S)-policy). An exception is the (T, Q)-policy of section 7.3, for which the inventory level is not bounded from above, but here very high inventory levels occur with very small probability. In case of a restrictive finite buffer, the optimal policy may change drastically.

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Samenvatting

Dit proefschrift betreft de wiskundige analyse van service systemen waarbij klanten binnen een vaste tijd (de "delay-limit") bediend moeten worden. De term klanten moet in brede zin opgevat worden; ook gedacht kan worden aan opdrachten, jobs in een computernetwerk, orders voor producten, etc. De modellen zijn stochastisch: de tussenaankomsttijden van klanten zijn niet op voorhand bekend en worden beschreven d.m.v. stochastische variabelen. Belangrijk is ook dat klanten geaggregeerd, d.w.z. tegelijk bediend, kunnen worden. Vanwege de vaste kosten die een groepsbediening met zich meebrengt, kunnen er schaalvoordelen behaald worden door zoveel mogelijk klanten te aggregeren; hoe groter het aantal klanten, hoe lager de kosten per klant. Echter, wanneer een klant zijn "delay-limit" bereikt terwijl het aantal wachtende klanten nog niet groot genoeg is om een groepsbediening te rechtvaardigen, dan moet deze klant individueel bediend worden. De kosten van een individuele bediening zijn beduidend hoger dan de variabele kosten van een groepsbediening (anders is het beter om *alle* klanten individueel te bedienen). Doelstelling is nu om voor deze situatie goede service strategieën te vinden, d.w.z. strategieën met zo laag mogelijke verwachte gemiddelde bedieningskosten per tijdseenheid, zodanig dat alle klanten binnen de "delay-limit" bediend worden.

Het proefschrift valt uiteen in twee delen. In Deel I beperken we ons tot het hierboven beschreven "service model", waarbij geen voorraad van de te leveren service kan worden aangelegd. In Hoofdstuk 2 beschouwen we een discrete-tijd variant waar bedieningen alleen op vaste intervallen gestart kunnen worden (bv. aan het eind van elke dag), en in Hoofdstuk 3 een continue-tijd variant waar bedieningen op elk willekeurig tijdstip mogelijk zijn. In Deel II generaliseren we het "service model" naar de productie-omgeving; klanten oefenen vraag uit naar een bepaald product dat op voorraad kan worden geproduceerd. De "delay-limit" bepaalt nu de maximale tijd gedurende welke een klant bereid is te wachten als er geen voorraad is, ofwel de maximale tijdspanne dat vraag kan worden nageleverd. In Hoofdstuk 4 introduceren we een algemeen raamwerk voor productie/voorraad modellen, met als belangrijkste parameters de "delay-limit", de productietijd, het aantal (identieke) machines en de capaciteit per machine. We nemen aan dat alle producten tegelijk beschikbaar komen aan het eind van de productietijd, zodat er een analogie ontstaat met voorraadmodellen waar besteld wordt bij een externe leverancier. Het blijkt cruciaal te zijn of de "delay-limit" kleiner dan wel groter is dan de productietijd. In het eerste geval kan er alleen maar vooruit worden geproduceerd, terwijl in het tweede geval eventueel gewacht kan worden tot de vraag gerealiseerd is. Hoofdstuk 5 heeft betrekking op het eerste geval, waarbij we ons beperken tot het geval van één machine met voldoende capaciteit. Dit leidt tot de welbekende strategieën voor stochastische voorraadmodellen. In Hoofdstuk 6 bekijken we het tweede geval, wederom zonder capaciteitsbeperkingen; hier zijn soortgelijke strategieën als in Deel I van toepassing met een extra variable voor de overproduktie. Vervolgens besteden we in Hoofdstuk 7 aandacht aan een algemener model met beperkingen op zowel het aantal machines als de capaciteit per machine, om inzicht te krijgen in de invloed van beperkte capaciteit. We besluiten het proefschrift in Hoofdstuk 8 met enkele algemene conclusies alsmede mogelijkheden voor verder onderzoek.

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Stellingen

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Service and Inventory Models subject to a Delay-Limit

van

Jorg Jansen

Katholieke Universiteit Brabant, 9 september 1998

I

Beschouw een M/D/∞ wachtrijmodel met constante service tijd C, en zij T_K de tijd totdat er voor het eerst K klanten in het systeem zijn (uitgaande van een leeg systeem). Dan wordt de verdelingsfunctie van T_K vastgelegd door

$$\Pr\{T_K > nC+t\} = e^{-\lambda(nC+t)} \sum_{\substack{l_i=0,\ldots,K-1-m_{i-1};\ i=1,\ldots,n+1\\ m_i=0,\ldots,K-1-l_i;\ i=1,\ldots,n}} (\lambda t)^{\sum_{i=1}^{n+1} l_i} \Big((\lambda(C-t)\Big)^{\sum_{i=1}^{n} m_i} \det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l))\det(C_n(\mathbf{a}^r,\mathbf{b}^r)),$$

voor $n = 0, 1, \ldots$ en $0 < t < C$, met $m_0 := 0$,

$$a_i^l := \sum_{j=1}^{i-1}(l_j + m_j) - (i-1)(K-1),\ b_i^l := \sum_{j=1}^{i}(m_{j-1} + l_j) - (i-1)(K-1) \quad (i = 1, \ldots, n+1),$$

$$a_i^r := \sum_{j=1}^{i}(m_{j-1} + l_j) - (i-1)(K-1),\ b_i^r := \sum_{j=1}^{i}(l_j + m_j) - (i-1)(K-1) \quad (i = 1, \ldots, n),$$

en

$$(C_n(\mathbf{a},\mathbf{b}))_{ij} = \begin{cases} \dfrac{1}{(b_i - a_j - i + j)!} & \text{als } b_i - a_j \geq i - j \\ 0 & \text{als } b_i - a_j < i - j \end{cases} \quad (i, j = 1, \ldots, n).$$

Zie: Dit proefschrift, Hoofdstuk 3.

II

Beschouw een voorraadmodel met periodieke inspectie, stochastische discrete vraag, "lost sales" en een constante positieve levertijd. Onder minimalisatie van verwachte lange-termijn gemiddelde bestel-, voorraad- en boetekosten per periode bestaat er een parameterinstelling en een voorraadniveau i, zodanig dat de optimale bestelhoeveelheid bij voorraadniveau i kleiner is dan bij voorraadniveau $i+1$.

Zie: Dit proefschrift, Hoofdstuk 5.

III

De kans dat de kandidaten in de finale van het televisiespelletje Lingo na n getrokken ballen af zijn ("Lingo" hebben), uitgaande van 35 blauwe ballen te trekken zonder teruglegging en 1 gouden bal te trekken met teruglegging, wordt gegeven door

$$\Pr\{\text{Lingo na } n \text{ trekkingen}\} = \sum_{k=\max\{n-35,0\}}^{n} p_k^{(n)} q_{n-k} \quad (n = 1, 2, \ldots),$$

waarbij

$$p_k^{(n)} := \Pr\{k \text{ gouden ballen in } n \text{ trekkingen}\} = \frac{36-n+k}{36} \sum_{\substack{i_1,\ldots,i_k:\\ 36-n+k\leq i_1\leq\cdots\leq i_k\leq 36}} \frac{1}{i_1\cdots i_k} \quad (k = 0, \ldots, n),$$

en

$$\begin{aligned} q_k &:= \Pr\{\text{Lingo na } k \text{ trekkingen} \mid \text{geen gouden ballen}\} \\ &= 9\,\frac{26!}{35!}\,\frac{(35-k)!}{(31-k)!}\,(k^4-118k^3+2867k^2+40930k+29400) \quad (k = 1, \ldots, 35). \end{aligned}$$

IV

Beschouw het dobbelspelletje Mexicaantje. Stel dat er n spelers zijn die allen de kans om "nat te gaan" minimaliseren. Dan hangt de optimale strategie van speler k ($2 \leq k \leq n$) af van het aantal worpen van speler 1, de laagste worp tot dan toe, het aantal spelers met de laagste worp tot dan toe, het aantal resterende spelers ($n-k$) en het aantal resterende worpen van speler k. De optimale strategie van speler 1 wordt volledig bepaald door $K_1^{(n)}$ en $K_2^{(n)}$, met $K_i^{(n)}$ de laagste worp waarop speler 1 moet passen met n spelers als er i worpen resteren ($i = 1, 2$). De waarden voor $K_1^{(n)}$ en $K_2^{(n)}$ voor $n = 2, \ldots, 10$ worden gegeven door:

n	2	3	4	5	6	7	8	9	10
$K_2^{(n)}$	54	52	52	51	43	43	43	43	42
$K_1^{(n)}$	61	54	53	53	52	52	52	52	51

V

Beschouw het vragenspel Triviant. Stel dat de kansen dat een speler een vraag goed beantwoordt zijn gegeven voor de zes verschillende categorieën, en dat de speler het aantal beurten tot de finish wil minimaliseren (onafhankelijk van de andere spelers). Dan is de optimale strategie voor deze speler een tabel die aangeeft naar welke kant het bakje verplaatst moet worden gegeven de worp van de dobbelsteen, de positie op het bord en de samenstelling van het bakje. De optimale strategie kan berekend worden middels een Markov beslissingsproces met 28032 ($= 6 \cdot 73 \cdot 2^6$) toestanden.

VI

Beschouw een badmintonwedstrijd tussen speler A en speler B over 2 gewonnen sets tot 15 punten, met bij 13-13 c.q. 14-14 de mogelijkheid om te "verlengen" tot 18 c.q. 17 voor de speler die als eerste 13 c.q. 14 heeft bereikt (service wisselt als bij volleybal). Stel dat speler A een willekeurige rally wint met kans p en verliest met kans $q := 1-p$. Dan moet speler A om de kans om de wedstrijd te winnen te maximaliseren:

- niet verlengen bij 13-13 noch bij 14-14 voor $0 < p \leq 0.4127$;
- niet verlengen bij 13-13 maar wel bij 14-14 voor $0.4127 < p \leq 0.4399$;
- zowel verlengen bij 13-13 als bij 14-14 voor $0.4399 < p < 1$.

Indien de alternatieve "five-to-nine" puntentelling gehanteerd wordt, d.w.z. 3 gewonnen sets tot 9 punten en bij 8-8 een mogelijke verlenging tot 11, dan moet speler A:

- niet verlengen bij 8-8 voor $0 < p \leq 0.4127$;
- wel verlengen bij 8-8 voor $0.4127 < p < 1$.

De "five-to-nine" puntentelling is voordelig voor de zwakkere speler, in de zin dat de kans dat speler A met $p < \frac{1}{2}$ de wedstrijd wint groter is onder de "five-to-nine" puntentelling dan onder de traditionele puntentelling.

VII

De hoeveelheid direct consumeerbare koffie in de Trie-angle tussen (zeg) 8 uur en 16 uur kan beschreven worden door een (s, Q) voorraadmodel met levertijd L en "lost sales", waarbij s correspondeert met de hoeveelheid koffie tot het streepje, Q met de hoeveelheid koffie in een volle pot en L met de benodigde tijd om een volle pot te zetten (onder de - wellicht niet realistische - veronderstelling dat ieder vakgroepslid zijn plicht doet). Het vraagproces kan beschreven worden door een compound Poisson proces $Y(t) = X(1) + \ldots + X(N(t))$ $(0 \leq t \leq 8)$, met $Y(t)$ de totale hoeveelheid gevraagde koffie tot $8+t$ uur, $N(t)$ het aantal binnengekomen groepen (van één of meer personen) tot $8+t$ uur en $X(i)$ de hoeveelheid gevraagde koffie door de i^e groep.

VIII

Het proefschrift van een AIO met een vage projectbeschrijving vertoont overeenkomsten met de toestand van een chaotisch dynamisch systeem op $t = 4$.

IX

Stelling IX behorende bij [Janssen 1998] gaat nergens over.
Zie: [Janssen 1998] Janssen, F.B.S.L.P. *Inventory Management Systems: control and information issues.* Proefschrift, Katholieke Universiteit Brabant, september 1998.

X

De vraag "Kun je nog iets anders zeggen dan ja?" is moeilijk overtuigend te beantwoorden.

XI

Voor het zoeken naar de inhoudsopgave van een boek (zoals een proefschrift) kan geen gebruik worden gemaakt van de inhoudsopgave.

XII

Het feit dat de achternaam "Huisman" wèl en "Huisvrouw" nìet voorkomt, zou erop kunnen duiden dat het beroep van huisman allerminst modern is.

XIII

Besliskundigen hebben vaak moeite met beslissen.

XIV

Ambitie ambiëren is ambitieus.

JORG JANSEN gradu
February 1994. After that he started working as a Ph.D. student at CentER, where he was a member of the Management Science group of the Department of Econometrics. His research was in the field of stochastic modelling and inventory control.

This thesis is concerned with the mathematical analysis of situations where service must be provided to customers within a prespecified time after arrival, the delay-limit (e.g., due to a service contract). Customer arrivals are governed by a stochastic process, and customers can be served jointly to obtain economies of scale. In Part I a basic model is extensively analysed, using techniques from Markov decision theory and queueing theory. In Part II this model is extended to the context of the production of exchangeable items, leading to a general framework for inventory models with a delay-limit on backorders. Several models within this framework are then studied in detail, including lost-sales inventory models.

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